

# THE DERIVED CATEGORY OF A GIT QUOTIENT

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**ABSTRACT.** Given a smooth algebraic variety with a reductive group action, I describe a relationship between its equivariant derived category and the derived category of its geometric invariant theory quotient. This generalizes classical descriptions of the category of coherent sheaves on projective space and categorifies several results in the theory of Hamiltonian group actions on projective manifolds.

This perspective generalizes and provides new insight into examples of derived equivalences between birational varieties. I provide a criterion under which two different GIT quotients are derived equivalent, and apply it to prove that any two generic GIT quotients of an equivariantly Calabi-Yau projective-over-affine variety by a torus are derived equivalent.

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## 1. INTRODUCTION

I will describe a relationship between the derived category of equivariant coherent sheaves on a smooth projective-over-affine variety,  $X$ , with an action of a reductive group,  $G$ , and the derived category of coherent sheaves on a GIT quotient of that action. The main theorem connects three classical circles of ideas:

- Serre's description of quasicoherent sheaves on a projective variety in terms of graded modules over its homogeneous coordinate ring,
- Kirwan's theorem that the canonical map  $H_G^*(X) \rightarrow H^*(X//G)$  is surjective,[13] and

- the “quantization commutes with reduction” theorem from geometric quantization theory equating  $h^0(X, \mathcal{L})^G$  with  $h^0(X//G, \mathcal{L})$  when the linearization  $\mathcal{L}$  descends to the GIT quotient.[17]

A  $G$ -linearized ample line bundle  $\mathcal{L}$  defines an open semistable locus  $X^{ss} \subset X$ , the complement of the base locus of invariant global sections of  $\mathcal{L}^k$  for  $k \gg 0$ . I will denote the quotient stack  $\mathfrak{X} = X/G$  and  $\mathfrak{X}^{ss} = X^{ss}/G$ . In this paper, I will refer to the quotient stack  $X^{ss}/G$  as the “GIT quotient”, even though that term more commonly refers to the coarse moduli space of  $X^{ss}/G$ .

In order to state the main theorem, I will need to recall the Kirwan-Ness (KN) stratification determined by a choice of  $G$ -invariant inner product on the compact form of  $\mathfrak{g}$ . [9] This is a  $G$ -equivariant stratification of  $X \setminus X^{ss}$  by connected locally-closed  $G$ -equivariant subvarieties  $\tilde{S}_\alpha$  indexed by a partially ordered set  $I$ . I will formally define a KN stratification in Definition 2.1, but for each  $\alpha$  there is a distinguished one parameter subgroup  $\lambda_\alpha : \mathbb{C}^* \rightarrow G$ , and a closed subvariety  $Z_\alpha \subset \tilde{S}_\alpha$  fixed by  $\lambda_\alpha$ .  $\eta_\alpha \geq 0$  will denote the sum of the  $\lambda_\alpha$  weights on the conormal sheaf  $\mathcal{I}_{S_\alpha}/\mathcal{I}_{S_\alpha}^2$  along  $Z_\alpha$ , and  $\kappa_\alpha : Z_\alpha \hookrightarrow X$  will denote the immersion. Because  $Z_\alpha$  is fixed by  $\lambda_\alpha$ , the restriction of an equivariant coherent sheaf  $\kappa_\alpha^* F$  is graded with respect to the weights of  $\lambda_\alpha$ .

I will denote the bounded derived category of coherent sheaves on  $\mathfrak{X}$  by  $D^b(\mathfrak{X})$ , and likewise for  $\mathfrak{X}^{ss}$ .<sup>1</sup> Restriction gives an exact dg-functor  $i^* : D^b(X/G) \rightarrow D^b(\mathfrak{X}^{ss}/G)$ , and in fact any bounded complex of equivariant coherent sheaves on  $X^{ss}$  can be extended equivariantly to  $X$ . The main result of this paper is the construction of a *functorial* splitting of  $i^*$ .

**Theorem 1.1** (derived Kirwan surjectivity, preliminary statement). *Let  $X$  be a smooth projective-over-affine variety with a linearized action of a reductive group  $G$ , and let  $\mathfrak{X} = X/G$ . Let  $q : I \rightarrow \mathbb{Z}$  be a function on the index set for the KN stratification of the unstable locus  $\mathfrak{X} \setminus \mathfrak{X}^{ss}$ . Define the full subcategory of  $D^b(\mathfrak{X})$*

$$\mathbf{G}_q := \left\{ F^\bullet \in D^b(\mathfrak{X}) \mid \mathcal{H}^*(L\kappa_\alpha^* F^\bullet) \text{ supported in weights } [q(\alpha), q(\alpha) + \eta_\alpha] \right\}$$

*Then the restriction functor  $i^* : \mathbf{G}_q \rightarrow D^b(\mathfrak{X}^{ss})$  is an equivalence of categories.*

**Remark 1.2.** I will prove a more complete version in Section 2 below. The full statement identifies  $\mathbf{G}_q$  as piece of a semiorthogonal decomposition of  $D^b(\mathfrak{X})$ , and applies to any open substack of a smooth stack  $\mathfrak{V} \subset \mathfrak{X}$  whose complement admits a KN stratification as in Definition 2.1.

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<sup>1</sup>On a technical note, all of the categories in this paper will be pre-triangulated dg-categories, so  $D^b(\mathfrak{X})$  denotes a dg-enhancement of the triangulated category usually denoted  $D^b(\mathfrak{X})$ . However, all of the results will be statements that can be verified on the level of homotopy categories, such as semiorthogonal decompositions and equivalences of categories, so I will often write proofs on the level of the underlying triangulated category.

Serre's theorem deals with the situation in which  $G = \mathbb{C}^*$ ,  $X$  is an affine cone, and the unstable locus consists only of the cone point – in other words one is studying a connected, positively graded  $k$ -algebra  $A$ . The category of quasicoherent sheaves on  $\mathrm{Proj}(A)$  can be identified with the full subcategory of the category of graded  $A$ -modules graded in degree  $\geq q$  for any fixed  $q$ . This classical result has been generalized to noncommutative  $A$  by M. Artin.[2] D. Orlov studied the derived categories and the category of singularities of such algebras in great detail in [15], and much of the technique of the proof of Theorem 1.1 derives from that paper.

The simplest example of Theorem 1.1 is familiar to many mathematicians: projective space  $\mathbb{P}(V)$  can be thought of as a GIT quotient of  $V/\mathbb{C}^*$ . Theorem 1.1 identifies  $D^b(\mathbb{P}(V))$  with the full triangulated subcategory of the derived category of equivariant sheaves on  $V$  generated by  $\mathcal{O}_V(q), \dots, \mathcal{O}_V(q + \dim V - 1)$ . In particular the semiorthogonal decompositions described in Section 3 refine and provide an alternative proof of Beilinson's theorem that the line bundles  $\mathcal{O}_{\mathbb{P}(V)}(1), \dots, \mathcal{O}_{\mathbb{P}(V)}(\dim V)$  generate  $D^b(\mathbb{P}(V))$ .

In the context of equivariant Kähler geometry, Theorem 1.1 is a categorification of Kirwan surjectivity. To be precise, one can recover the De Rham cohomology of a smooth stack as the periodic-cyclic homology its derived category[12, 19], so the classical Kirwan surjectivity theorem follows from the existence of a splitting of  $i^*$ . Kirwan surjectivity applies to topological  $K$ -theory as well[10], and one immediate corollary of Theorem 1.1 is an analogous statement for algebraic  $K$ -theory

**Corollary 1.3.** *The restriction map on algebraic  $K$ -theory  $K_i(\mathfrak{X}) \rightarrow K_i(\mathfrak{X}^{ss})$  is surjective.*

The fully faithful embedding  $D^b(\mathfrak{X}^{ss}) \subset D^b(\mathfrak{X})$  of Theorem 1.1 and the more precise semiorthogonal decomposition of Theorem 2.2 correspond, via Orlov's analogy between derived categories and motives[14], to the claim that the motive  $\mathfrak{X}^{ss}$  is a summand of  $\mathfrak{X}$ . Via this analogy, the results of this paper bear a strong formal resemblance to the motivic direct sum decompositions of homogeneous spaces arising from Białynicki-Birula decompositions[6]. However, the precise analogue of Theorem 1.1 would pertain to the equivariant motive  $X/G$ , whereas the results of [6] pertain to the nonequivariant motive  $X$ .

The “quantization commutes with reduction” theorem from geometric quantization theory relates to the fully-faithfulness of the functor  $i^*$ . The original conjecture of Guillemin and Sternberg, that  $\dim H^0(X/G, \mathcal{L}^k) = \dim H^0(X^{ss}/G, \mathcal{L}^k)$ , has been proven by several authors, but the most general version was proven by Teleman in [17]. He shows that the canonical restriction map induces an isomorphism  $R\Gamma(X/G, \mathcal{V}) \rightarrow R\Gamma(X^{ss}/G, \mathcal{V})$  for any equivariant vector bundle such that  $\mathcal{V}|_{Z_\alpha}$  is supported in weight  $> -\eta_\alpha$ . If  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are two vector bundles in the grade restriction windows of Theorem 1.1, then the fact that  $R\mathrm{Hom}_{\mathfrak{X}}^\bullet(\mathcal{V}_1, \mathcal{V}_2) \rightarrow R\mathrm{Hom}_{\mathfrak{X}^{ss}}^\bullet(\mathcal{V}_1|_{\mathfrak{X}^{ss}}, \mathcal{V}_2|_{\mathfrak{X}^{ss}})$

is an isomorphism is precisely Teleman's quantization theorem applied to  $\mathcal{V}_2 \otimes \mathcal{V}_1^\vee \simeq R\mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ .

The main application of Theorem 1.1 in this paper is to construct equivalences between the derived categories of different GIT quotients of  $X/G$ . The  $G$ -ample cone of  $X$  has a decomposition into convex conical chambers[9] within which the GIT quotient  $\mathfrak{X}^{ss}(\mathcal{L})$  does not change. I present a criterion under which the GIT quotients on either side of a wall separating two chambers are derived equivalent, and give several applications. The wall crossing criterion gives an explicit description of how the derived category changes under a birational cobordism. It also implies that any two generic torus quotients of an equivariantly Calabi-Yau variety are derived equivalent.

I apply the wall crossing criterion to flops which excise a subvariety isomorphic to a weighted-projective or Grassmannian bundle over a smooth variety and replace it with the dual Grassmannian bundle. Derived equivalences for flops of Grassmannians were recently investigated by Will Donovan in his PhD thesis,[8] and the connection with grade restriction windows will be described in a forthcoming paper of Donovan and Ed Segal. The advantage of grade restriction windows is that they render all of the above derived equivalences tautological – both quotients are identified by Theorem 1.1 with the *same* subcategory of  $D^b(X/G)$ .

Finally, Section 5 extends Theorem 2.2 and its applications to categories of matrix factorizations and the the derived categories of some complete intersections and hyperkähler reductions. The argument is purely a formal consequence of derived Morita theory.[3]

The inspiration for Theorem 1.1 were the grade restriction rules for the category of B-branes of Landau-Ginzburg models studied by Hori, Herbst, and Page,[11] as interpreted mathematically by Segal.[16]. The essential idea of splitting was present in that paper, but the analysis was only carried out for a linear action of  $\mathbb{C}^*$ , and the category  $\mathbf{G}_q$  was identified in an ad-hoc way. The main contribution of this paper is showing that the splitting can be globalized and applies to arbitrary  $X/G$  as a categorification of Kirwan surjectivity, and that the categories  $\mathbf{G}_q$  arise naturally via the semiorthogonal decompositions to be described in the next section.

I would like to thank my PhD adviser Constantin Teleman for introducing me to his work [17], and for his support and useful comments throughout this project. I would like to thank Daniel Pomerleano for years of enlightening conversations, and specifically for patiently explaining how to recover derived categories of singularities using Morita theory. I'd like to thank Anatoly Preygel for useful conversations about derived algebraic geometry and for carefully reviewing section 5. Finally I'd like to thank Yujiro Kawamata for suggesting that I apply my methods to hyperkähler reduction and flops of Grassmannian bundles.

The problems studied in this paper overlap greatly with the forthcoming work [4], although the projects were independently conceived and carried out. I learned about [4] at the January 2012 Conference on Homological

Mirror Symmetry at the University of Miami, where the authors presented a method of constructing derived equivalences between toric varieties. My understanding is that since that time they have extended their results to include  $\mathbb{C}^*$  actions on smooth quasiprojective varieties and some instances of reductive groups acting on affine space. However, this paper focuses on the global splitting using grade restriction windows, whereas they produce derived equivalences using subcategories of a resolution of a flop. We hope that the different perspectives brought to bear on these issues will be useful in elucidating further questions.

## 2. THE MAIN THEOREM

In this section we will review the geometric structure of the KN stratification and lay out the proof of Theorem 1.1. By a simple inductive argument one can reduce the proof to the construction of a splitting of  $D^b(\mathfrak{X}) \rightarrow D^b(\mathfrak{X} - \mathfrak{S})$  for a single closed KN stratum  $\mathfrak{S} \subset \mathfrak{X}$ , a stratum of maximal codimension in  $X$ . Of course there is no a-priori reason for restriction to an open substack to admit a splitting. We will use the rich geometric structure of  $\mathfrak{S}$  that arises because  $\mathfrak{S}$  is a KN stratum of a GIT stratification:

For a maximal stratum, the geometric situation is a smooth quasiprojective  $X$  with a  $G$ -action, an equivariant ample line bundle  $\mathcal{L}$ , and a smooth closed  $G$  invariant subscheme  $j : \tilde{S} \hookrightarrow X$ .<sup>2</sup> We denote the complement  $V = X - \tilde{S}$ . We will denote the quotients of these schemes by  $G$  by  $\mathfrak{X}$ ,  $\mathfrak{S}$ , and  $\mathfrak{V}$  respectively.

From GIT theory there is a distinguished 1-PS  $\lambda : \mathbb{C}^* \rightarrow G$  associated to  $\mathfrak{S}$ . As usual this defines the parabolic subgroup  $P \subset G$  of all  $p \in G$  such that  $\lambda(t)p\lambda(t)^{-1}$  has a limit as  $t \rightarrow 0$ . In addition we will define  $L \subset P$  to be the commutant of  $\lambda$ , it is a Levi component of  $P$ , so we have the semidirect product sequence

$$1 \longrightarrow U \longrightarrow P \overset{\leftarrow}{\rightleftarrows} L \longrightarrow 1$$

where  $U \subset P$  is the unipotent radical. We will suppress the dependence of  $U, P$ , and  $L$  on  $\lambda$  from the notation, as  $\lambda$  will be fixed throughout this section (although different GIT strata have different  $\lambda$ ).

The subscheme  $\tilde{S}$  has some special properties with respect to  $\lambda$

(S1) There is a smooth closed subscheme  $S \subset X$  invariant under  $P$  such that  $\tilde{S} = G \cdot S$ , and the canonical map  $G \times_P S \rightarrow \tilde{S}$  is an isomorphism.

(S2) If we define  $Z = S \cap X^{\lambda \text{ fixed}}$ , then  $S$  consists entirely of those points in  $X$  whose limit  $\lim_{t \rightarrow 0} \lambda(t) \cdot x \in Z$ . This map  $\pi : x \mapsto \lim_{t \rightarrow 0} \lambda(t) \cdot x$  gives  $S$  the structure of a  $P$  equivariant bundle of affine spaces over  $Z$ . Thus  $S = \underline{\text{Spec}}_Z(\mathcal{A})$  where  $\mathcal{A} = \mathcal{O}_Z \oplus \bigoplus_{i < 0} \mathcal{A}_i$  is a  $P$ -equivariant

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<sup>2</sup>We use the notation  $\tilde{S}$  for the  $G$ -equivariant stratum and reserve the notation  $S$  for a more central object of study.

$\mathcal{O}_Z$  algebra which is locally isomorphic to a polynomial algebra over  $\mathcal{O}_Z$ .  $\mathcal{A}$  is supported in non-positive weights with respect to  $\lambda$ .

(S3) The restriction of the ideal sheaf  $\mathcal{I}_{\tilde{\mathfrak{S}}} \subset \mathcal{O}_X$  to  $Z$  has positive weights with respect to  $\lambda$ .

The conormal sheaf  $\Omega_{\mathfrak{S}/\mathfrak{X}} = \mathcal{I}_{\mathfrak{S}}/\mathcal{I}_{\mathfrak{S}}^2$  is locally free on  $\mathfrak{S}$ , and by (S3) the restriction of  $\det(\Omega_{\mathfrak{S}/\mathfrak{X}})$  to  $\mathfrak{Z}$  is concentrated in a single nonnegative weight with respect to  $\lambda$  (it is 0 iff  $\Omega_{\mathfrak{S}/\mathfrak{X}} = 0$ ). This weight number  $\eta$ , the sum of the weights occurring in  $\Omega_{\mathfrak{S}/\mathfrak{X}}$ , will determine the width of the grade restriction windows.

As the statement of Theorem 1.1 indicates, we will construct a splitting of  $D^b(\mathfrak{X}) \rightarrow D^b(\mathfrak{Y})$  by identifying, for any  $w \in \mathbb{Z}$ , a subcategory  $\mathbf{G}_w \subset D^b(\mathfrak{X})$  that is mapped isomorphically onto  $D^b(\mathfrak{Y})$ . In fact we will identify  $\mathbf{G}_w$  by first describing a “baric structure” structure on  $D_{\mathfrak{S}}^b(\mathfrak{X})$ , the bounded derived category of complexes of coherent sheaves on  $\mathfrak{X}$  whose homology is supported on  $\mathfrak{S}$ . First we must recall some common notions from homological algebra.

We denote a *semiorthogonal decomposition* of a triangulated category  $\mathcal{D}$  by full triangulated subcategories  $\mathcal{A}_i$  as  $\mathcal{D} = \langle \mathcal{A}_n, \dots, \mathcal{A}_1 \rangle$ .<sup>[5]</sup> This means that all morphisms from objects in  $\mathcal{A}_i$  to objects in  $\mathcal{A}_j$  are zero for  $i < j$ , and for any object of  $E \in \mathcal{D}$  there is a sequence  $0 = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n = E$  with  $\text{Cone}(E_{i-1} \rightarrow E_i) \in \mathcal{A}_i$ , which is necessarily unique and thus functorial.<sup>3</sup> In our applications  $\mathcal{D}$  will always be a pre-triangulated category, in which case if  $\mathcal{A}_i \subset \mathcal{D}$  are full pre-triangulated categories then we will abuse the notation  $\mathcal{D} = \langle \mathcal{A}_n, \dots, \mathcal{A}_1 \rangle$  to mean that there is a semiorthogonal decomposition of homotopy categories, in which case  $\mathcal{D}$  is uniquely identified with the gluing of the  $\mathcal{A}_i$ .<sup>[?]</sup>

A *baric decomposition* is simply a filtration of a triangulated category  $\mathcal{D}$  by right-admissible triangulated subcategories, i.e. a family of semiorthogonal decompositions  $\mathcal{D} = \langle \mathcal{D}_{<w}, \mathcal{D}_{\geq w} \rangle$  such that  $\mathcal{D}_{\geq w} \supset \mathcal{D}_{\geq w+1}$ , and thus  $\mathcal{D}_{<w} \subset \mathcal{D}_{<w+1}$ , for all  $w$ . For categories of coherent sheaves which have (derived) tensor products and inner Hom’s, we will say that the baric decomposition is *multiplicative* if  $\mathcal{D}_{\geq v} \otimes \mathcal{D}_{\geq w} \subset \mathcal{D}_{\geq v+w}$  or equivalently  $R\text{Hom}(\mathcal{D}_{\geq v}, \mathcal{D}_{<w}) \subset \mathcal{D}_{<w-v}$ .

Baric decompositions were originally used to construct ‘staggered’  $t$ -structures on equivariant derived categories generalizing those used to define perverse coherent sheaves. Although the connection with GIT was not explored in the original development of the theory, baric decompositions seem to be the natural structure arising on the derived category of the unstable locus in geometric invariant theory.

<sup>3</sup>There are two additional equivalent ways to characterize a semiorthogonal decomposition: 1) the inclusion of the full subcategory  $\mathcal{A}_i \subset \langle \mathcal{A}_i, \mathcal{A}_{i-1}, \dots, \mathcal{A}_1 \rangle$  admits a left adjoint  $\forall i$ , or 2) the subcategory  $\mathcal{A}_i \subset \langle \mathcal{A}_n, \dots, \mathcal{A}_i \rangle$  is right admissible  $\forall i$ . In some contexts one also requires that each  $\mathcal{A}_i$  be admissible in  $\mathcal{D}$ , but we will not require this here. See [5] for further discussion.

For a closed KN stratum  $\mathfrak{S} \subset \mathfrak{X}$ , let  $\kappa : \mathfrak{Z} \hookrightarrow \mathfrak{X}$  denote the closed substack described in Property (S2). In Proposition 3.19 we will show the existence of a multiplicative baric decomposition  $D_{\mathfrak{S}}^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$  where

$$D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} := \{F^\bullet | \mathcal{H}^*(L\kappa^*F^\bullet) \text{ supported in weight } \geq w \text{ w.r.t } \lambda\}$$

$$D_{\mathfrak{S}}^b(\mathfrak{X})_{<w} := \{F^\bullet | \mathcal{H}^*(L\kappa^*F^\bullet) \text{ supported in weight } < w + \eta \text{ w.r.t } \lambda\}$$

Furthermore, in Theorem 3.21 we will extend this baric decomposition to a semiorthogonal decomposition of the whole derived category  $D^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}, \mathbf{G}_w, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$  where

$$\mathbf{G}_w = \{F^\bullet \in D^b(\mathfrak{X}) | \lambda \text{ weights of } \mathcal{H}^*(L\kappa^*F^\bullet) \text{ lie in } [w, w + \eta]\}$$

And the restriction functor  $i^* : \mathbf{G}_w \rightarrow D^b(\mathfrak{Y})$  is an equivalence. The inverse is given by choosing an extension of  $G^\bullet \in D^b(\mathfrak{Y})$  to  $F^\bullet \in D^b(\mathfrak{X})$  and then taking  $\text{Cone}(F_1^\bullet \rightarrow F_2^\bullet) \in \mathbf{G}_w$ , where  $0 = F_0^\bullet \rightarrow F_1^\bullet \rightarrow F_2^\bullet \rightarrow F_3^\bullet = F^\bullet$  is the canonical factorization coming from the semiorthogonal decomposition.

The construction of the semiorthogonal decompositions of Theorem 3.21 and Proposition 3.19 is part of a larger story about the derived categories of  $\mathfrak{Z}$  and  $\mathfrak{S}$ , so we will postpone further discussion to Section 3. However, just knowing the existence of these semiorthogonal decompositions will be sufficient for the rest of this section.

**Definition 2.1** (KN stratification). A stratification of  $\mathfrak{X}^u \subset \mathfrak{X}$  by locally closed substacks  $\mathfrak{S}_\alpha \subset \mathfrak{X}^u$  indexed by a partially ordered set  $I$  will be called a Kirwan-Ness (KN) stratification if

- (1) For any  $\alpha$ , the union of the strata  $\bigcup_{\alpha' \geq \alpha} \mathfrak{S}_{\alpha'}$  is closed in  $\mathfrak{X}$ .
- (2) Each  $\mathfrak{S}_\alpha$  satisfies (S1), (S2), and (S3) in  $\mathfrak{X} \setminus \bigcup_{\alpha' > \alpha} \mathfrak{S}_{\alpha'}$ .

For a KN stratification, we will let  $\kappa_\alpha : \mathfrak{Z}_\alpha \hookrightarrow X$  be the immersion of the closed substacks  $\mathfrak{Z}_\alpha \subset \mathfrak{S}_\alpha$  described in Property (S2), and we will let  $\lambda_\alpha$  denote the distinguished 1-PS for each stratum. We will always have a KN stratification on the unstable locus of a projective GIT quotient.[9]

**Theorem 2.2** (derived Kirwan surjectivity). *Let  $\mathfrak{X}$  be a smooth stack, let  $\mathfrak{X}^{ss} \subset \mathfrak{X}$  be an open substack, and let  $\{\mathfrak{S}_\alpha\}_{\alpha \in I}$  be a KN stratification (Definition 2.1) of  $\mathfrak{X}^u = \mathfrak{X} \setminus \mathfrak{X}^{ss}$ .*

*For any  $q : I \rightarrow \mathbb{Z}$ , define the full subcategories of  $D^b(\mathfrak{X})$*

$$\mathbf{G}_q := \{F^\bullet | \forall \alpha \in I, \lambda_\alpha \text{ weights of } \mathcal{H}^*(L\kappa_\alpha^*F^\bullet) \text{ lie in } [w_\alpha, w_\alpha + \eta_\alpha]\}$$

$$D_{\mathfrak{X}^u}^b(\mathfrak{X})_{\geq q} := \{F^\bullet \in D_{\mathfrak{X}^u}^b(\mathfrak{X}) | \forall \alpha \in I, \lambda_\alpha \text{ weights of } \mathcal{H}^*(L\kappa_\alpha^*F^\bullet) \text{ are } \geq q(\alpha)\}$$

$$D_{\mathfrak{X}^u}^b(\mathfrak{X})_{< q} := \{F^\bullet \in D_{\mathfrak{X}^u}^b(\mathfrak{X}) | \forall \alpha \in I, \lambda_\alpha \text{ weights of } \mathcal{H}^*(L\kappa_\alpha^*F^\bullet) \text{ are } < q(\alpha) + \eta_\alpha\}$$

*Then there are semiorthogonal decompositions*

$$D_{\mathfrak{X}^u}^b(\mathfrak{X}) = \langle D_{\mathfrak{X}^u}^b(\mathfrak{X})_{< q}, D_{\mathfrak{X}^u}^b(\mathfrak{X})_{\geq q} \rangle \quad (1)$$

$$D^b(\mathfrak{X}) = \langle D_{\mathfrak{X}^u}^b(\mathfrak{X})_{< q}, \mathbf{G}_q, D_{\mathfrak{X}^u}^b(\mathfrak{X})_{\geq q} \rangle \quad (2)$$

and the restriction functor  $i^* : \mathbf{G}_q \rightarrow D^b(\mathfrak{X}^{ss})$  is an equivalence of categories. We have  $D_{\mathfrak{X}^u}^b(\mathfrak{X})_{\geq q_1} \otimes^L D_{\mathfrak{X}^u}^b(\mathfrak{X})_{\geq q_2} \subset D_{\mathfrak{X}^u}^b(\mathfrak{X})_{\geq q_1+q_2}$ .

**Remark 2.3.** All of the results of Section 3 are local to the strata, so this theorem holds for  $\mathfrak{X}$  which are only smooth in a neighborhood of  $\mathfrak{X}^u$ .

*Proof.* Choose a total ordering of  $I$ ,  $\alpha_0 > \alpha_1 > \dots$  such that  $\alpha_n$  is maximal in  $I \setminus \{\alpha_0, \dots, \alpha_{n-1}\}$ , so that  $\mathfrak{S}_{\alpha_n}$  is closed in  $\mathfrak{X} \setminus \mathfrak{S}_{\alpha_0} \cup \dots \cup \mathfrak{S}_{\alpha_{n-1}}$ . Introduce the notation  $\mathfrak{S}^n = \bigcup_{i < n} \mathfrak{S}_{\alpha_i}$ .  $\mathfrak{S}^n \subset \mathfrak{X}$  is closed and admits a KN stratification by the  $n$  strata  $\mathfrak{S}_{\alpha_i}$  for  $i < n$ , so we will proceed by induction on  $n$ . The base case is Theorem 3.21.

Assume the theorem holds for  $\mathfrak{S}^n \subset \mathfrak{X}$ , so  $D^b(\mathfrak{X}) = \langle D_{\mathfrak{S}^n}^b(\mathfrak{X})_{< q}, \mathbf{G}_q^n, D_{\mathfrak{S}^n}^b(\mathfrak{X})_{\geq q} \rangle$  and restriction maps  $\mathbf{G}_q^n$  isomorphically onto  $D^b(\mathfrak{X} \setminus \mathfrak{S}^n)$ .  $\mathfrak{S}_{\alpha_n} \subset \mathfrak{X} \setminus \mathfrak{S}^n$  is a closed KN stratum, so Theorem 3.21 gives a semiorthogonal decomposition of  $\mathbf{G}_q^n \simeq D^b(\mathfrak{X} \setminus \mathfrak{S}^n)$  which we combine with the previous semiorthogonal decomposition

$$D^b(\mathfrak{X}) = \langle D_{\mathfrak{S}^n}^b(\mathfrak{X})_{< q}, D_{\mathfrak{S}_{\alpha_n}}^b(\mathfrak{X} \setminus \mathfrak{S}^n)_{< q(\alpha)}, \mathbf{G}_q^{n+1}, D_{\mathfrak{S}_{\alpha_n}}^b(\mathfrak{X} \setminus \mathfrak{S}^n)_{\geq q(\alpha)}, D_{\mathfrak{S}^n}^b(\mathfrak{X})_{\geq q} \rangle$$

The first two pieces correspond precisely to  $D_{\mathfrak{S}^{n+1}}^b(\mathfrak{X})_{< q}$  and the last two pieces correspond to  $D_{\mathfrak{S}^{n+1}}^b(\mathfrak{X})_{\geq q}$ . The theorem follows by induction.  $\square$

Given an  $F^\bullet \in D^b(\mathfrak{X}^{ss})$ , one can extend it uniquely up to weak equivalence to a complex in  $\mathbf{G}_q$ . Due to the inductive nature of Theorem 2.2, the extension can be complicated to construct. We will discuss a procedure for extending over a single stratum at the end of Section 3, and one must repeat this for every stratum of  $\mathfrak{X}^{us}$ .

Fortunately, it suffices to directly construct a single universal extension. Consider the product  $\mathfrak{X}^{ss} \times \mathfrak{X} = (X^{ss} \times X)/(G \times G)$ , and the open substack  $\mathfrak{X}^{ss} \times \mathfrak{X}^{ss}$  whose complement has the KN stratification  $\mathfrak{X}^{ss} \times \mathfrak{S}_\alpha$ . One can uniquely extend the diagonal  $\mathcal{O}_{\mathfrak{X}^{ss} \times \mathfrak{X}^{ss}}$  to a sheaf  $\tilde{\mathcal{O}}_\Delta$  in the subcategory  $\mathbf{G}_q$  with respect to this stratification. The Fourier-Mukai transform  $D^b(\mathfrak{X}^{ss}) \rightarrow D^b(\mathfrak{X})$  with kernel  $\tilde{\mathcal{O}}_\Delta$ , has image in the subcategory  $\mathbf{G}_q$  and is the identity over  $\mathfrak{X}^{ss}$ . Thus for any  $F^\bullet \in D^b(\mathfrak{X}^{ss})$ ,  $(p_2)_*(\tilde{\mathcal{O}}_\Delta \otimes p_1^*(F^\bullet))$  is the unique extension of  $F^\bullet$  to  $\mathbf{G}_q$ .

### 3. HOMOLOGICAL STRUCTURES ON THE UNSTABLE STRATA

In this section we will study in detail the homological properties of a single closed KN stratum, i.e. a closed substack  $\mathfrak{S} \subset \mathfrak{X}$  satisfying properties (S1)-(S3). We establish a multiplicative baric decomposition of  $D^b(\mathfrak{Z})$ , and we extend it first to a multiplicative baric decomposition of  $D^b(\mathfrak{S})$ , and then finally to  $D_{\mathfrak{S}}^b(\mathfrak{X})$ , the derived category of complexes of coherent sheaves on  $\mathfrak{X}$  whose restriction to  $\mathfrak{Y} = \mathfrak{X} - \mathfrak{S}$  is acyclic.

We will use the phrase  $\mathcal{O}_Z$ -module to denote a quasicoherent sheaf on the stack  $\mathfrak{Z} = Z/P$ , assuming quasicoherence and  $P$ -equivariance unless otherwise specified.  $\lambda$  fixes  $Z$ , so equivariant  $\mathcal{O}_Z$  modules have a natural grading by the weight spaces of  $\lambda$ , and we will use this grading often.



**Remark 3.1.** The results of this subsection do not require any smoothness hypotheses on  $Z$

**Lemma 3.2.** *For any  $F \in \mathrm{QCoh}(\mathfrak{Z})$  and any  $w \in \mathbb{Z}$ , the submodule  $F_{\geq w} := \sum_{i \geq w} F_i$  of sections of weight  $\geq w$  with respect to  $\lambda$  is  $P$  equivariant.*

*Proof.*  $\mathbb{C}^*$  commutes with  $L$ , so  $F_{\geq w}$  is an equivariant submodule with respect to the  $L$  action. Because  $U \subset P$  acts trivially on  $Z$ , the  $U$ -equivariant structure on  $F$  is determined by a coaction  $a : F \rightarrow k[U] \otimes F$  which is equivariant for the  $\mathbb{C}^*$  action. We have

$$a(F_{\geq w}) \subset (k[U] \otimes F)_{\geq w} = \bigoplus_{i+j \geq w} k[U]_i \otimes F_j \subset k[U] \otimes F_{\geq w}$$

The last inclusion is due to the fact that  $k[U]$  is non-positively graded, and it implies that  $F_{\geq w}$  is equivariant with respect to the  $U$  action as well. Because we have a semidirect product decomposition  $P = UL$ , it follows that  $F_{\geq w}$  is an equivariant submodule with respect to the  $P$  action.  $\square$

**Remark 3.3.** This lemma is a global version of the observation that for any  $P$ -module  $M$ , the subspace  $M_{\geq w}$  with weights  $\geq w$  with respect to  $\lambda$  is a  $P$ -submodule, which can be seen from the coaction  $M \rightarrow k[P] \otimes M$  and the fact that  $k[P]$  is nonnegatively graded with respect to  $\lambda$ .

It follows that any  $F \in \mathrm{QCoh}(\mathfrak{Z})$  has a functorial factorization  $F_{\geq w} \hookrightarrow F \twoheadrightarrow F_{< w}$ . Note that as  $\mathbb{C}^*$ -equivariant instead of  $P$ -equivariant  $\mathcal{O}_Z$ -modules there is a natural isomorphism  $F \simeq F_{\geq w} \oplus F_{< w}$ . Thus the functors  $(\bullet)_{\geq w}$  and  $(\bullet)_{< w}$  are exact, and that if  $F$  is locally free, then  $F_{\geq w}$  and  $F_{< w}$  are locally free as well.

We define  $\mathrm{QCoh}(\mathfrak{Z})_{\geq w}$  and  $\mathrm{QCoh}(\mathfrak{Z})_{< w}$  to be the full subcategories of  $\mathrm{QCoh}(\mathfrak{Z})$  consisting of sheaves supported in weight  $\geq w$  and weight  $< w$  respectively. They are both Serre subcategories, they are orthogonal to one another,  $(\bullet)_{\geq w}$  is right adjoint to the inclusion  $\mathrm{QCoh}(\mathfrak{Z})_{\geq w} \subset \mathrm{QCoh}(\mathfrak{Z})$ , and  $(\bullet)_{< w}$  is left adjoint to the inclusion  $\mathrm{QCoh}(\mathfrak{Z})_{< w} \subset \mathrm{QCoh}(\mathfrak{Z})$ .

**Lemma 3.4.** *Any  $F \in \mathrm{QCoh}(\mathfrak{S})_{< w}$  admits an injective resolution  $F \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \cdots$  such that  $\mathcal{I}^i \in \mathrm{QCoh}(\mathfrak{Z})_{< w}$ . Likewise any  $F \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$  admits a locally free resolution  $\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow F$  such that  $E_i \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$ .*

*Proof.* First assume  $F \in \mathrm{QCoh}(\mathfrak{Z})_{< w}$ , and let  $F \rightarrow \mathcal{I}^0$  be the injective hull of  $F$ .<sup>4</sup> Then  $\mathcal{I}_{\geq w}^0 \cap F_{< w} = 0$ , hence  $\mathcal{I}_{\geq w}^0 = 0$  because  $\mathcal{I}^0$  is an essential extension of  $F$ .  $\mathrm{QCoh}(\mathfrak{Z})_{< w}$  is a Serre subcategory, so  $\mathcal{I}^0/F \in \mathrm{QCoh}(\mathfrak{Z})_{< w}$  as well, and we can inductively build an injective resolution with  $\mathcal{I}^i \in \mathrm{QCoh}(\mathfrak{Z})_{< w}$ .

Next assume  $F \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$ . Choose a surjection  $E \rightarrow F$  where  $E$  is locally free. Then  $E_0 := E_{\geq w}$  is still locally free, and  $E_{\geq w} \rightarrow F$  is still surjective. Because  $\mathrm{Coh}(\mathfrak{Z})_{\geq w}$  is a Serre subcategory,  $\ker(E_0 \rightarrow F) \in$

<sup>4</sup>The injective hull exists because  $\mathrm{QCoh}(\mathfrak{Z})$  is cocomplete and taking filtered colimits is exact.

$\mathrm{Coh}(\mathfrak{Z})_{\geq w}$  as well, so we can inductively build a locally free resolution with  $E_i \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$ .  $\square$

We will use this lemma to study the subcategories of  $D^b(\mathfrak{Z})$  generated by  $\mathrm{Coh}(\mathfrak{Z})_{\geq w}$  and  $\mathrm{Coh}(\mathfrak{Z})_{< w}$ . Define the full triangulated subcategories

$$\begin{aligned} D^b(\mathfrak{Z})_{\geq w} &= \{F^\bullet \in D^b(\mathfrak{Z}) \mid \mathcal{H}^i(F^\bullet) \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}\} \\ D^b(\mathfrak{Z})_{< w} &= \{F^\bullet \in D^b(\mathfrak{Z}) \mid \mathcal{H}^i(F^\bullet) \in \mathrm{Coh}(\mathfrak{Z})_{< w}\} \end{aligned}$$

For any complex  $F^\bullet$  we have the canonical short exact sequence

$$0 \rightarrow F_{\geq w}^\bullet \rightarrow F^\bullet \rightarrow F_{< w}^\bullet \rightarrow 0 \quad (3)$$

If  $F^\bullet \in D^b(\mathfrak{Z})_{\geq w}$  then the first arrow is a quasi-isomorphism, because  $(\bullet)_{\geq w}$  is exact. Likewise for the second arrow if  $F^\bullet \in D^b(\mathfrak{Z})_{< w}$ . Thus  $F^\bullet \in D^b(\mathfrak{Z})_{\geq w}$  iff it is quasi-isomorphic to a complex of sheaves in  $\mathrm{Coh}(\mathfrak{Z})_{\geq w}$  and likewise for  $D^b(\mathfrak{Z})_{< w}$ .

**Proposition 3.5.** *These subcategories constitute a multiplicative baric decomposition  $D^b(\mathfrak{Z}) = \langle D^b(\mathfrak{Z})_{< w}, D^b(\mathfrak{Z})_{\geq w} \rangle$ . This baric decomposition is bounded, meaning that every object lies in  $\mathcal{D}_{\geq w} \cap \mathcal{D}_{< v}$  for some  $w, v$ . The baric truncation functors, the adjoints of the inclusions  $\mathcal{D}_{\geq w}, \mathcal{D}_{< w} \subset D^b(\mathfrak{Z})$ , are exact.*

*Proof.* If  $A \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$  and  $B \in \mathrm{Coh}(\mathfrak{Z})_{< w}$ , then by Lemma 3.4 we resolve  $B$  by injectives in  $\mathrm{QCoh}(\mathfrak{Z})_{< w}$ , and thus  $R\mathrm{Hom}(A, B) \simeq 0$ . It follows that  $D^b(\mathfrak{Z})_{\geq w}$  is left orthogonal to  $D^b(\mathfrak{Z})_{< w}$ . One proves that  $D^b(\mathfrak{Z})_{\geq w} \otimes D^b(\mathfrak{Z})_{\geq v} \subset D^b(\mathfrak{Z})_{\geq v+w}$  by resolving by locally frees in  $\mathrm{Coh}(\mathfrak{Z})_{\geq w}$  and  $\mathrm{Coh}(\mathfrak{Z})_{\geq v}$ .

$\mathrm{QCoh}(\mathfrak{Z})_{\geq w}$  and  $\mathrm{QCoh}(\mathfrak{Z})_{\leq w}$  are Serre subcategories, so  $F_{\geq w}^\bullet \in D^b(\mathfrak{Z})_{\geq w}$  and  $F_{< w}^\bullet \in D^b(\mathfrak{Z})_{< w}$  for any  $F^\bullet \in D^b(\mathfrak{Z})$ . Thus the natural sequence (3) shows that we have a baric decomposition, and that the right and left truncation functors are the exact functors  $(\bullet)_{\geq w}$  and  $(\bullet)_{< w}$  respectively. Boundedness follows from the fact that coherent equivariant  $\mathcal{O}_Z$ -modules must be supported in finitely many  $\lambda$  weights.  $\square$

**3.1. Quasicoherent sheaves on  $\mathfrak{S}$ .** Next we will construct a baric decomposition of  $D^b(\mathfrak{S})$ . We begin by naively mimicking the definition of  $\mathrm{QCoh}(\mathfrak{Z})_{\geq, < w}$ , but we will have to define  $D^b(\mathfrak{S})_{\geq w}$  differently than  $D^b(\mathfrak{Z})_{\geq w}$ .

As mentioned in property (S2),  $\mathfrak{S} \simeq \underline{\mathrm{Spec}}_Z(\mathcal{A})/P$ , where  $\mathcal{A}$  is a locally polynomial  $\mathcal{O}_Z$ -module with  $\mathcal{A}_i = 0$  for  $i > 0$ , and  $\mathcal{A}_0 = \mathcal{O}_Z$ . We will identify quasicoherent sheaves on the quotient stack  $\mathfrak{S} = S/P$  with  $P$ -equivariant quasicoherent  $\mathcal{A}$ -modules on  $Z$ .

For any  $F \in \mathrm{QCoh}(\mathfrak{S})$  we can consider  $F_{\geq w}$  as an equivariant  $\mathcal{O}_S$ -submodule of  $F$ . Define the  $\mathcal{A}$  submodule generated in weights  $\geq w$  as  $\beta_{\geq w}F := \mathcal{A} \cdot F_{\geq w}$ . It is automatically  $P$ -equivariant, and we define  $\beta_{< w}$  by the functorial short exact sequence

$$0 \rightarrow \beta_{\geq w}F \rightarrow F \rightarrow \beta_{< w}F \rightarrow 0 \quad (4)$$

$(\beta_{<w}F)_i = 0$  for  $i \geq w$ , but because  $\mathcal{A}$  has negative weight spaces,  $(\beta_{\geq w}F)_i$  will typically be non-zero for arbitrarily negative  $i$ .

We define two full subcategories of  $\mathrm{QCoh}(\mathfrak{S})$

$$\begin{aligned} \mathrm{QCoh}(\mathfrak{S})_{\geq w} &= \{F \text{ such that } \beta_{\geq w}F \rightarrow F \text{ is surjective}\} \\ \mathrm{QCoh}(\mathfrak{S})_{<w} &= \{F \text{ such that } F_i = 0 \text{ for } i \geq w\} \\ &= \{F \text{ such that } F \rightarrow \beta_{<w}F \text{ is injective}\} \end{aligned}$$

An  $F \in \mathrm{QCoh}(\mathfrak{S})_{\geq w}$  is said to be *generated in weight  $\geq w$* , and an  $F \in \mathrm{QCoh}(\mathfrak{S})_{<w}$  is said to be *supported in weight  $< w$* . The functor  $\beta_{\geq w}$  is the right adjoint of the inclusion  $\mathrm{QCoh}(\mathfrak{S})_{\geq w} \subset \mathrm{QCoh}(\mathfrak{S})$ , and  $\beta_{<w}$  is the left adjoint of the inclusion  $\mathrm{QCoh}(\mathfrak{S})_{<w} \subset \mathrm{QCoh}(\mathfrak{S})$ .

Unlike the category  $\mathrm{QCoh}(\mathfrak{Z})_{\geq w}$ , the category  $\mathrm{QCoh}(\mathfrak{S})_{\geq w}$  is not closed under subobjects. A simple example is the ideal  $\mathcal{A}_{<0} \subset \mathcal{A}$  defining  $Z \subset S$  – it is clearly not generated in weight  $\geq 0$ . However we do have the following

**Lemma 3.6.**  *$\mathrm{QCoh}(\mathfrak{S})_{<w}$  is a Serre subcategory of  $\mathrm{QCoh}(\mathfrak{S})$ . The subcategory  $\mathrm{QCoh}(\mathfrak{S})_{\geq w}$  is closed under quotients and extensions*

*Proof.* The first claim is evident from the definition of  $\mathrm{QCoh}(\mathfrak{S})_{<w}$  by forgetting the  $\mathcal{A}$ -module structure and using the fact that the weights under  $\lambda$  give functorial direct sum decompositions.

Given a short exact sequence of equivariant  $\mathcal{A}$ -modules  $0 \rightarrow F'' \rightarrow F \rightarrow F' \rightarrow 0$ , we get an exact sequence of  $\mathcal{O}_Z$ -modules  $0 \rightarrow F''_{\geq w} \rightarrow F_{\geq w} \rightarrow F'_{\geq w} \rightarrow 0$ . Thus if  $F_{\geq w}$  generates  $F$ , then  $F'_{\geq w}$  generates  $F'$ . Also, if  $\beta_{\geq w}F'' = F''$  and  $\beta_{\geq w}F' = F'$ , then  $\beta_{\geq w}F$  contains  $F''$  and maps surjectively onto  $F'$ , so  $\beta_{\geq w}F = F$ .  $\square$

In order to characterize elements of  $\mathrm{Coh}(\mathfrak{S})_{\geq w}$  we prove

**Lemma 3.7.** *Let  $F \in \mathrm{Coh}(\mathfrak{S})$ , then the following are equivalent*

- (1)  $F \in \mathrm{Coh}(\mathfrak{S})_{\geq w}$ ,
- (2) *there exists a surjection  $\mathcal{A} \otimes_{\mathcal{O}_Z} E \twoheadrightarrow F$  where  $E \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$  is locally free, and*
- (3)  $F \otimes_{\mathcal{A}} \mathcal{O}_Z \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$ .

**Remark 3.8.** Because  $\mathrm{Coh}(\mathfrak{S})_{\geq w}$  is not closed under subobjects, this lemma does not guarantee the existence of a left resolution by locally free sheaves in  $\mathrm{Coh}(\mathfrak{S})_{\geq w}$ .

*Proof.* 1  $\Rightarrow$  2: By the ascending chain condition there is a coherent  $\mathcal{O}_Z$ -submodule  $F' \subset F_{\geq w}$  which generates  $F$ . We choose a surjection  $E \rightarrow F'$  from a locally free sheaf in  $\mathrm{Coh}(\mathfrak{Z})_{\geq w}$ , and then  $\mathcal{A} \otimes E \rightarrow F$  is surjective.

2  $\Rightarrow$  3: If  $\mathcal{A} \otimes E \rightarrow F$  is surjective, then taking the tensor product with  $\mathcal{O}_Z$  gives a surjection  $E \rightarrow F \otimes \mathcal{O}_Z$ . Hence  $F \otimes \mathcal{O}_Z \in \mathrm{Coh}(\mathfrak{Z})_{\geq w}$ .

3  $\Rightarrow$  1: Choose a surjection  $\mathcal{A} \otimes E \rightarrow F$  where  $E \in \mathrm{Coh}(\mathfrak{Z})$  is locally free. By hypothesis  $\mathcal{A} \otimes E_{\geq w} \rightarrow F$  is surjective after tensoring with  $\mathcal{O}_Z$ , so it

follows from Nakayama's lemma that  $\mathcal{A} \otimes E_{\geq w} \rightarrow F$  is surjective.<sup>5</sup> The image of  $E_{\geq w}$  is contained in  $F_{\geq w}$  and generates  $F$ , hence  $F \in \text{Coh}(\mathfrak{S})_{\geq w}$ .  $\square$

If  $F \in \text{QCoh}(\mathfrak{S})_{\geq w}$  then the surjection  $\mathcal{A} \otimes F_{\geq w} \rightarrow F$  gives  $\text{Hom}_{\mathfrak{S}}(F, G) \hookrightarrow \text{Hom}_{\mathfrak{S}}(\mathcal{A} \otimes F_{\geq w}, G) = \text{Hom}_{\mathfrak{Z}}(F_{\geq w}, G)$ . The last group is 0 if  $G \in \text{QCoh}(\mathfrak{S})_{< w}$  because then  $F_{\geq w}$  and  $G$  are graded in non-overlapping weights with respect to  $\lambda$ . A similar argument using a surjection  $\mathcal{A} \otimes E_{\geq w} \rightarrow F$  shows that if  $F$  is coherent and  $G \in \text{QCoh}(\mathfrak{S})_{< v}$  then  $\underline{\text{Hom}}_{\mathfrak{S}}(F, G) \in \text{QCoh}(\mathfrak{S})_{< v-w}$ .

Unfortunately the categories  $\text{Coh}(\mathfrak{S})_{\geq w}$  and  $\text{Coh}(\mathfrak{S})_{< w}$  are no longer orthogonal in  $\text{D}^b(\mathfrak{S})$ . The example mentioned above  $0 \rightarrow \mathcal{A}_{< 0} \rightarrow \mathcal{A} \rightarrow \mathcal{O}_Z \rightarrow 0$  corresponds to a nontrivial class in  $\text{Ext}_{\mathfrak{S}}^1(\mathcal{O}_Z, \mathcal{A}_{< 0})$ . However, we have

**Lemma 3.9.** *Let  $E \in \text{QCoh}(\mathfrak{Z})_{\geq w}$ , and let  $F^\bullet$  be bounded below with  $H^i(F^\bullet) \in \text{QCoh}(\mathfrak{S})_{< w}$  for all  $i$ . Then  $R\text{Hom}_{\mathfrak{S}}(\mathcal{A} \otimes E, F^\bullet) = 0$ .*

*Proof.* Let  $\pi : \mathfrak{S} \rightarrow \mathfrak{Z}$  be the projection. Then  $\mathcal{A} \otimes E \simeq L\pi^*E$  because  $\mathcal{A}$  is flat over  $\mathcal{O}_Z$ . From the derived adjunction we have  $R\text{Hom}_{\mathfrak{S}}(L\pi^*E, F^\bullet) \simeq R\text{Hom}_{\mathfrak{Z}}(E, R\pi_*F^\bullet)$ .  $\pi$  is affine, so  $R\pi_*F^\bullet \simeq \pi_*F^\bullet \in \text{D}^+(\mathfrak{Z})_{< w}$ . The claim follows from the fact that  $\text{QCoh}(\mathfrak{Z})_{\geq w}$  is left orthogonal to  $\text{D}^+(\mathfrak{Z})_{< w}$ .  $\square$

Thus we will focus on subcategory of  $\text{D}^b(\mathfrak{S})$  generated by sheaves of the form  $\mathcal{A} \otimes E$  with  $E \in \text{Coh}(\mathfrak{Z})_{\geq w}$ . We define

$$\begin{aligned} \text{D}^b(\mathfrak{S})_{< w} &= \{F^\bullet \in \text{D}^b(\mathfrak{S}) \mid \mathcal{H}^i(F^\bullet) \in \text{QCoh}(\mathfrak{S})_{< w} \text{ for all } i\} \\ \text{D}^b(\mathfrak{S})_{\geq w} &= \{F^\bullet \in \text{D}^b(\mathfrak{S}) \mid F^\bullet \simeq \mathcal{A} \otimes E^\bullet \text{ with } E^i \in \text{Coh}(\mathfrak{Z})_{\geq w}\} \end{aligned}$$

Note that  $\text{D}^b(\mathfrak{S})_{< w}$  is a thick triangulated subcategory because  $\text{QCoh}(\mathfrak{S})_{< w}$  is a Serre subcategory of  $\text{QCoh}(\mathfrak{S})$ . We will show in Proposition 3.13 that  $\text{D}^b(\mathfrak{S})_{\geq w}$  is a thick triangulated category as well.

Consider a complex in  $\text{QCoh}(\mathfrak{S})$  of the form  $\mathcal{A} \otimes E^\bullet$ . Note that the differential  $\mathcal{A} \otimes E^i \rightarrow \mathcal{A} \otimes E^{i+1}$  is not necessarily induced from a differential  $E^i \rightarrow E^{i+1}$ . However we observe

**Lemma 3.10.** *If  $E \in \text{QCoh}(\mathfrak{Z})$ , then  $\beta_{\geq w}(\mathcal{A} \otimes E) \simeq \mathcal{A} \otimes E_{\geq w}$ .*

*Proof.*  $\beta_{\geq w}(\mathcal{A} \otimes E)$  is the largest  $\mathcal{A}$ -submodule of  $\mathcal{A} \otimes E$  generated in weight  $\geq w$ , so  $\mathcal{A} \otimes E_{\geq w} \subset \beta_{\geq w}(\mathcal{A} \otimes E)$ . Conversely  $\beta_{\geq w}(\mathcal{A} \otimes E)$  is the  $\mathcal{A}$ -submodule generated by  $\bigoplus_{i+j \geq w} \mathcal{A}_i \otimes E_j$ , which is the same as the  $\mathcal{A}$ -submodule generated by  $\bigoplus_{j \geq w} \mathcal{A}_0 \otimes E_j \subset \mathcal{A} \otimes E_{\geq w}$ . Thus  $\beta_{\geq w}(\mathcal{A} \otimes E) \subset \mathcal{A} \otimes E_{\geq w}$ .  $\square$

This guarantees that  $\mathcal{A} \otimes E_{\geq w}$  is a subcomplex, and  $E_{\geq w}$  is a direct summand as a non-equivariant  $\mathcal{O}_Z$ -module, so we have a canonical short exact sequence of complexes in  $\text{QCoh}(\mathfrak{S})$

$$0 \rightarrow \mathcal{A} \otimes E_{\geq w}^\bullet \rightarrow \mathcal{A} \otimes E^\bullet \rightarrow \mathcal{A} \otimes E_{< w}^\bullet \rightarrow 0 \quad (5)$$

<sup>5</sup>The natural extension of Nakayama's lemma to stacks is the statement that the support of a coherent sheaf is closed. In our setting we consider an  $F \in \text{Coh}(\mathfrak{S})$ . If  $F \otimes \mathcal{O}_Z = 0$ , then  $\text{supp}(F) \cap Z = \emptyset$ , but every nonempty closed substack of  $S$  intersects  $Z$  nontrivially, so  $F = 0$ . We can apply this to the cokernel of a map of coherent sheaves to conclude that the map is surjective iff it is surjective after tensoring with  $\mathcal{O}_Z$ .

Using this sequence we prove

**Proposition 3.11.** *The categories  $D^b(\mathfrak{S}) = \langle D^b(\mathfrak{S})_{<w}, D^b(\mathfrak{S})_{\geq w} \rangle$  constitute a multiplicative baric decomposition.*

*Proof.* Note that every  $F \in \text{Coh}(\mathfrak{S})$  has a highest weight space as an equivariant  $\mathcal{O}_Z$ -module  $F_{\geq h} \neq 0$  where  $F_{\geq w} = 0$  for  $w > h$ . Furthermore, because  $\mathcal{A}_{<0} \in \text{QCoh}(\mathfrak{Z})_{<0}$  the map  $(F)_{\geq h} \rightarrow (F \otimes \mathcal{O}_Z)_{\geq h}$  is an isomorphism.

In fact any coherent equivariant  $\mathcal{O}_Z$ -submodule  $E^0 \subset F$  which generates  $F$  as an equivariant  $\mathcal{A}$ -module must contain  $F_{\geq h}$ . It follows that  $\ker(\mathcal{A} \otimes E^0 \rightarrow F)$  has highest weight strictly less than  $h$ . Iterating this construction we get a resolution  $\cdots \rightarrow \mathcal{A} \otimes E^{-1} \rightarrow \mathcal{A} \otimes E^0 \rightarrow F$  such that for any  $w$ ,  $E_{\geq w}^i = 0$  for  $i \ll 0$ . We can likewise construct such a presentation for any  $F^\bullet \in D^b(\mathfrak{S})$ .

The canonical short exact sequence (5) gives an exact triangle  $\mathcal{A} \otimes E_{\geq w}^\bullet \rightarrow F^\bullet \rightarrow \mathcal{A} \otimes E_{<w}^\bullet \dashrightarrow$ . The first term is in  $D^b(\mathfrak{S})_{\geq w}$ , and because both the first and second term have bounded cohomology, the third term has bounded cohomology as well, and it is manifestly in  $D^b(\mathfrak{S})_{<w}$ . Finally, Lemma 3.9 implies that  $R\text{Hom}(F^\bullet, G^\bullet) = 0$  for  $F^\bullet \in D^b(\mathfrak{S})_{\geq w}$  and  $G^\bullet \in D^b(\mathfrak{S})_{<w}$ . Thus we have our weak semiorthogonal decomposition.

The multiplicativity of  $D^b(\mathfrak{S})_{\geq w}$  follows from the fact that  $D^b(\mathfrak{Z})_{\geq w}$  is multiplicative, the fact that  $L\pi^*$  respects derived tensor products, the fact that  $\mathcal{A} \otimes E \simeq L\pi^*E$  for  $E \in \text{QCoh}(\mathfrak{Z})$ , and the fact that sheaves of the form  $\mathcal{A} \otimes E$  generate  $D^b(\mathfrak{S})_{\geq w}$ .  $\square$

In the remainder of this subsection we will identify a more intrinsic characterization of  $D^b(\mathfrak{S})_{\geq w}$ . Let  $\sigma : \mathfrak{Z} \hookrightarrow \mathfrak{S}$  be the canonical section of  $\mathfrak{S} \rightarrow \mathfrak{Z}$ . First we observe the following extension of Nakayama's lemma to the derived category

**Lemma 3.12** (Nakayama). *Let  $F^\bullet \in D^-(\mathfrak{S})$  with coherent cohomology. If  $L\sigma^*F^\bullet \simeq 0$ , then  $F^\bullet \simeq 0$ .*

*Proof.* If  $H^r(F^\bullet)$  is the highest nonvanishing cohomology group, then  $H^r(L\sigma^*F^\bullet) \simeq \sigma^*H^r(F^\bullet)$ . By Nakayama's lemma  $\sigma^*H^r(F^\bullet) = 0 \Rightarrow H^r(F^\bullet) = 0$ , so we must have  $\sigma^*H^r(F^\bullet) \neq 0$  as well.  $\square$

From the definition in Proposition 3.11 it is not even clear that  $D^b(\mathfrak{S})_{\geq w}$  is a triangulated subcategory. However, using the derived version of Nakayama's lemma we have

**Proposition 3.13.** *Let  $F^\bullet \in D^b(\mathfrak{S})$ . Then the following are equivalent*

- (1)  $L\sigma^*F^\bullet \in D^b(\mathfrak{Z})_{\geq w}$ ,
- (2)  $F^\bullet$  is quasi-isomorphic to a right-bounded complex of the sheaves of the form  $\mathcal{A} \otimes E^i$  with  $E^i \in \text{Coh}(\mathfrak{Z})_{\geq w}$  locally free,
- (3)  $F^\bullet \in D^b(\mathfrak{S})_{\geq w}$ .

*Proof.*  $1 \Leftrightarrow 2$ : Only one direction requires proof. Let  $L\sigma^*F^\bullet \in D^b(\mathfrak{Z})_{\geq w}$ . We choose a right bounded presentation by locally frees  $\mathcal{A} \otimes E^\bullet \simeq F^\bullet$  and consider the canonical sequence (5).

Restricting to  $\mathfrak{Z}$  gives a short exact sequence  $0 \rightarrow E_{\geq w}^\bullet \rightarrow E^\bullet \rightarrow E_{<w}^\bullet \rightarrow 0$ . The first and second terms have homology in  $\text{Coh}(\mathfrak{Z})_{\geq w}$ , and the third has homology in  $\text{Coh}(\mathfrak{Z})_{<w}$ . These two categories are orthogonal, so it follows from the long exact homology sequence that  $E_{<w}^\bullet$  is acyclic. Thus by Nakayama's lemma  $\mathcal{A} \otimes E_{<w}^\bullet$  is acyclic and  $F^\bullet \simeq \mathcal{A} \otimes E_{\geq w}^\bullet$ .

$1 \Leftrightarrow 3$ : For any  $E \in \text{Coh}(\mathfrak{Z})$ ,  $L\sigma^*(\mathcal{A} \otimes E) \simeq E$ , so if  $F^\bullet$  is equivalent to a finite complex of the form  $\mathcal{A} \otimes E_{\geq w}^\bullet$ , then  $L\sigma^*F^\bullet \in D^b(\mathfrak{Z})_{\geq w}$ .

Conversely if  $L\sigma^*F^\bullet \in D^b(\mathfrak{Z})_{\geq w}$ , consider the factorization from the proof of Proposition 3.11,  $\mathcal{A} \otimes E_{\geq w}^\bullet \hookrightarrow \mathcal{A} \otimes E^\bullet \twoheadrightarrow \mathcal{A} \otimes E_{<w}^\bullet$ , where the left term is a bounded complex and the middle term is a right-bounded complex equivalent to  $F^\bullet$ . Applying  $L\sigma^*$  to this sequence,  $L\sigma^*(\mathcal{A} \otimes E_{<w}^\bullet) \in D^b(\mathfrak{Z})_{<w}$  is acyclic by the same reasoning as above. Hence  $\mathcal{A} \otimes E_{<w}^\bullet$  is acyclic by Nakayama's lemma and we get  $\mathcal{A} \otimes E_{\geq w}^\bullet \simeq F^\bullet$ .  $\square$

**Remark 3.14.** By a slightly more refined construction, relying on the smoothness of  $\mathfrak{Z}$ , one can always choose a presentation  $\mathcal{A} \otimes E^\bullet \simeq F^\bullet$  by locally frees such that for each  $w$ ,  $E_{\geq w}^i = 0$  for  $i \ll 0$ . It follows that  $\beta_{\geq w}F^\bullet$  is equivalent to a *finite* complex of locally frees of the form  $\mathcal{A} \otimes E_{\geq w}^i$ .

We note the role of Serre duality on  $\mathfrak{S}$ . Let  $\omega_{\mathfrak{S}} := (\bigwedge^{\text{top}} \mathfrak{g}) \otimes (\bigwedge^{\text{top}} \Omega_S^1)$  be the equivariant canonical bundle of  $\mathfrak{S}$ . It is a dualizing bundle on  $\mathfrak{S}$  and we define the Serre duality functor  $\mathbb{D}_{\mathfrak{S}}(\bullet) = R\text{Hom}(\bullet, \omega_{\mathfrak{S}})$ . From the discussion above, we have seen that  $F^\bullet \in D^b(\mathfrak{S})_{\geq w}$  iff it admits a resolution by locally free  $\mathcal{A}$ -modules whose weights along  $\mathfrak{Z}$  are  $\geq w$ . Conversely,  $F^\bullet \in D^b(\mathfrak{S})_{<w}$  iff it admits a finite presentation by vector bundles whose weights along  $\mathfrak{Z}$  are  $< w$ . It follows that

$$\mathbb{D}_{\mathfrak{S}}(D^b(\mathfrak{S})_{\geq w}) = D^b(\mathfrak{S})_{< a+1-w}$$

where  $a$  is the weight of  $\lambda$  on the fiber of  $\omega_{\mathfrak{S}}$  along  $\mathfrak{Z}$ . Because  $\mathcal{A}$  is concentrated in nonpositive weights, we have  $a \leq 0$ , with equality iff  $\mathcal{A} = \mathcal{O}_Z$ , i.e. the entire stratum is fixed by  $\lambda$ .

**3.2. Quasicoherent sheaves with support on  $\mathfrak{S}$ .** Now we return to the derived category of  $\mathfrak{X}$  itself. We will extend the baric decomposition of  $D^b(\mathfrak{S})$  to a baric structure of  $D_{\mathfrak{S}}^b(\mathfrak{X})$ . Then we will construct the semiorthogonal decomposition

$$D^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}, \mathbf{G}_w, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$$

such that  $i^* : \mathbf{G}_w \rightarrow D^b(\mathfrak{Y})$  is an equivalence.

**Definition 3.15.** We define the thick triangulated subcategories of  $D^b(\mathfrak{X})$

$$\begin{aligned} D^b(\mathfrak{X})_{\geq w} &:= \{F^\bullet \in D^b(\mathfrak{X}) \mid Lj^*F^\bullet \in D^b(\mathfrak{S})_{\geq w}\} \\ D^b(\mathfrak{X})_{<w} &:= \{F^\bullet \in D^b(\mathfrak{X}) \mid Rj^!F^\bullet \in D^b(\mathfrak{S})_{<w}\} \end{aligned}$$

These do not form a baric decomposition on  $\mathfrak{X}$ , but they are important for the following reason

**Theorem 3.16** (Quantization Theorem). *Let  $F^\bullet \in D^b(\mathfrak{X})_{\geq w}$  and  $G^\bullet \in D^b(\mathfrak{X})_{<v}$  and  $w \geq v$ , then the restriction map*

$$R \operatorname{Hom}_{\mathfrak{X}}(F^\bullet, G^\bullet) \rightarrow R \operatorname{Hom}_{\mathfrak{Y}}(F^\bullet|_{\mathfrak{Y}}, G^\bullet|_{\mathfrak{Y}})$$

*is an isomorphism.*

*Proof.* This is equivalent to the vanishing of  $R\Gamma_{\mathfrak{S}}(R\operatorname{Hom}_{\mathfrak{X}}(F^\bullet, G^\bullet))$ . By the formula  $Rj^! \operatorname{Hom}_{\mathfrak{X}}(F^\bullet, G^\bullet) \simeq \operatorname{Hom}_{\mathfrak{S}}(Lj^* F^\bullet, Rj^! G^\bullet)$  it suffices to prove the case where  $F^\bullet = \mathcal{O}_X$ , i.e. showing that  $R\Gamma_{\mathfrak{S}}(G^\bullet) = 0$  whenever  $Rj^! G^\bullet \in D(\mathfrak{S})_{<0}$ .

We have

$$R\Gamma_{\mathfrak{S}}(G^\bullet) = \varinjlim R \operatorname{Hom}_{\mathfrak{X}}(\mathcal{O}_X / \mathcal{I}_{\mathfrak{S}}^n, G^\bullet)$$

so it suffices to show the vanishing of each term in the limit. From the short exact sequences of  $\mathcal{O}_X$  modules

$$0 \rightarrow \mathcal{I}_{\mathfrak{S}}^n / \mathcal{I}_{\mathfrak{S}}^p \rightarrow \mathcal{I}_{\mathfrak{S}}^m / \mathcal{I}_{\mathfrak{S}}^p \rightarrow \mathcal{I}_{\mathfrak{S}}^m / \mathcal{I}_{\mathfrak{S}}^n \rightarrow 0 \text{ for all } m < n < p \quad (6)$$

we see that  $\mathcal{O}_X / \mathcal{I}_{\mathfrak{S}}^n$  is in the triangulated subcategory generated by  $\mathcal{I}_{\mathfrak{S}}^n / \mathcal{I}_{\mathfrak{S}}^{n+1} = j_* \operatorname{Sym}^n(\Omega_{\mathfrak{S}} \mathfrak{X})$  where  $\Omega_{\mathfrak{S}} \mathfrak{X} = \mathcal{I}_{\mathfrak{S}} / \mathcal{I}_{\mathfrak{S}}^2$  is the conormal sheaf.

Thus we have reduced the claim to showing that  $R \operatorname{Hom}(j_* \operatorname{Sym}^n(\Omega_{\mathfrak{S}} \mathfrak{X}), G^\bullet) \simeq R \operatorname{Hom}_{\mathfrak{X}}(\operatorname{Sym}^n(\Omega_{\mathfrak{S}} \mathfrak{X}), Rj^! G^\bullet) = 0$ . By hypothesis  $Rj^! G^\bullet \in D(\mathfrak{S})_{<0}$  and  $\operatorname{Sym}^n \Omega_{\mathfrak{S}} \mathfrak{X} \in D^b(\mathfrak{S})_{\geq 0}$ , so the vanishing follows from the baric decomposition on  $\mathfrak{S}$  of Proposition 3.11.  $\square$

**Remark 3.17.** The key fact for the quantization theorem is that the conormal cone  $\bigoplus \mathcal{I}_{\mathfrak{S}} / \mathcal{I}_{\mathfrak{S}}^2$  lies in  $D^b(\mathfrak{S})_{\geq 0}$

**Lemma 3.18.** *Let  $\mathcal{I}_{\mathfrak{S}}$  be the ideal sheaf of  $\mathfrak{S}$  and let  $V$  be a locally free sheaf on  $\mathfrak{S}$ . Then  $\operatorname{Tor}_{\mathcal{O}_{\mathfrak{X}}}^i(\mathcal{I}_{\mathfrak{S}}, j_* V) \simeq j_* V \otimes \bigwedge^{i+1}(\mathcal{I}_{\mathfrak{S}} / \mathcal{I}_{\mathfrak{S}}^2)$ .*

Using this well known lemma we can uniquely extend the baric decomposition on  $D^b(\mathfrak{S})$  to  $D_{\mathfrak{S}}^b(\mathfrak{X})$ .

**Proposition 3.19.** *There is a unique multiplicative baric decomposition  $D_{\mathfrak{S}}^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$  such that*

$$j_*(D^b(\mathfrak{S})_{\geq w}) \subset D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \text{ and } j_*(D^b(\mathfrak{S})_{<w}) \subset D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$$

*It is described explicitly by*

$$\begin{aligned} D_{\mathfrak{S}}^b(\mathfrak{X})_{<w} &= \{F^\bullet \in D_{\mathfrak{S}}^b(\mathfrak{X}) \mid Rj^! F^\bullet \in D^b(\mathfrak{S})_{<w}\} \\ D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} &= \{F^\bullet \in D_{\mathfrak{S}}^b(\mathfrak{X}) \mid Lj^* F^\bullet \in D^b(\mathfrak{S})_{\geq w}\} \end{aligned}$$

*Proof.* Note that a necessary condition for this proposition to hold is that  $j_*(D^b(\mathfrak{S})_{\geq w})$  is right orthogonal to  $j_*(D^b(\mathfrak{S})_{<w})$ , which is equivalent to showing  $Lj^* j_*(D^b(\mathfrak{S})_{\geq w}) \subset D^b(\mathfrak{S})_{\geq w}$ . It suffices to check this for locally free  $V \in D^b(\mathfrak{S})_{\geq w}$  because they generate. By Lemma 3.18,  $H^i(Lj^* j_* V) \simeq V \otimes \bigwedge^{i+1} j^* \mathcal{I}_{\mathfrak{S}} \in D^b(\mathfrak{S})_{\geq w}$ , so  $Lj^* j_* V \in D^b(\mathfrak{S})_{\geq w}$ .

Now let  $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}$  and  $D_{\mathfrak{S}}^b(\mathfrak{X})_{< w}$  be the thick triangulated subcategories generated by  $j_*(D_{\mathfrak{S}}^b(\mathfrak{S})_{\geq w})$  and  $j_*(D_{\mathfrak{S}}^b(\mathfrak{S})_{< w})$  respectively.  $D_{\mathfrak{S}}^b(\mathfrak{X})_{< w}$  is right orthogonal to  $D_{\mathfrak{S}}^b(\mathfrak{S})_{\geq w}$ , so what we must show is that  $D_{\mathfrak{S}}^b(\mathfrak{X}) = D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \star D_{\mathfrak{S}}^b(\mathfrak{X})_{< w}$ , where the  $\mathcal{A} \star \mathcal{B}$  denotes the full subcategory consisting of  $F$  admitting triangles  $A \rightarrow F \rightarrow B \dashrightarrow$  with  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ .

It is an exercise in triangulated categories that if  $\mathcal{A}$  and  $\mathcal{B}$  are triangulated subcategories, and  $\mathcal{B} \subset \mathcal{A}^\perp$ , then the subcategory  $\mathcal{A} \star \mathcal{B}$  is triangulated as well. Furthermore, for any  $F \in D^b(\mathfrak{S})$  we have the exact triangle  $j_*\beta_{\geq w}F \rightarrow j_*F \rightarrow j_*\beta_{< w}F \dashrightarrow$ , so  $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \star D_{\mathfrak{S}}^b(\mathfrak{X})_{< w}$  is a triangulated subcategory containing  $j_*(D^b(\mathfrak{S}))$ , and so  $D_{\mathfrak{S}}^b(\mathfrak{X}) = D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \star D_{\mathfrak{S}}^b(\mathfrak{X})_{< w}$  as desired.

Now that we have shown that  $D_{\mathfrak{S}}^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{< w}, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$ , we have

$$\begin{aligned} F^\bullet \in D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} &\Leftrightarrow R\mathrm{Hom}_{\mathfrak{X}}(F^\bullet, j_*G^\bullet) = 0, \quad \forall G^\bullet \in D^b(\mathfrak{S})_{< w} \\ &\Leftrightarrow R\mathrm{Hom}_{\mathfrak{X}}(Lj^*F^\bullet, G^\bullet) = 0, \quad \forall G^\bullet \in D^b(\mathfrak{S})_{< w} \\ &\Leftrightarrow Lj^*F^\bullet \in D^b(\mathfrak{S})_{\geq w} \end{aligned}$$

A similar computation shows that  $F^\bullet \in D_{\mathfrak{S}}^b(\mathfrak{X})_{< w}$  iff  $Rj^!F^\bullet \in D^b(\mathfrak{S})_{< w}$ .  $\square$

The baric decomposition of Proposition 3.19 extends uniquely to a baric decomposition of the derived category of quasicoherent sheaves

$$D_{\mathfrak{S}}(\mathfrak{X}) = \langle D_{\mathfrak{S}}(\mathfrak{X})_{< w}, D_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \rangle \quad (7)$$

By the following general fact: if  $\mathcal{C} \subset \mathcal{T}$  is a triangulated subcategory which generates  $\mathcal{T}$  and consists of compact objects, and if  $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ , then  $\mathcal{T} = \langle \bar{\mathcal{A}}, \bar{\mathcal{B}} \rangle$ . To prove this note that  $\bar{\mathcal{B}}$  is right admissible by Brown-Neeman representability, and  $\mathcal{A}$  generates  $\bar{\mathcal{B}}^\perp$  because  $\mathcal{A}$  and  $\mathcal{B}$  together generate  $\mathcal{T}$ .

The last ingredient we need is the following

**Lemma 3.20.** *Let  $F^\bullet \in D^b(\mathfrak{X})$ . Then for sufficiently large  $N$  the canonical map*

$$\beta_{\geq w} R\mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_X/\mathcal{I}_{\mathfrak{S}}^N, F^\bullet) \rightarrow \beta_{\geq w} R\Gamma_{\mathfrak{S}}(F^\bullet)$$

*is an equivalence, hence  $\beta_{\geq w} R\Gamma_{\mathfrak{S}} : D^b(\mathfrak{X}) \rightarrow D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}$ .*

*Proof.*  $R\Gamma_{\mathfrak{S}}(F^\bullet) = \varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_X/\mathcal{I}_{\mathfrak{S}}^n, I^\bullet)$  for a homotopy injective replacement  $F^\bullet \rightarrow I^\bullet$ . Thus the short exact sequence (6) gives a short exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_X/\mathcal{I}_{\mathfrak{S}}^N, I^\bullet) \rightarrow \varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_X/\mathcal{I}_{\mathfrak{S}}^n, I^\bullet) \rightarrow \varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathcal{I}_{\mathfrak{S}}^N/\mathcal{I}_{\mathfrak{S}}^n, I^\bullet) \rightarrow 0$$

$\mathcal{I}_{\mathfrak{S}}^n/\mathcal{I}_{\mathfrak{S}}^{n+1}$  is a vector bundle along  $\mathfrak{S}$  generated in weights  $\geq n$ , so due to the short exact sequence (6) we have  $\mathcal{I}_{\mathfrak{S}}^N/\mathcal{I}_{\mathfrak{S}}^n \in D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq N}$  for all  $n > N$ .

Now  $Rj^!F^\bullet \in D^b(\mathfrak{S})_{< W}$  for some sufficiently large  $W$ , so  $R\mathrm{Hom}_{\mathfrak{X}}(\mathcal{I}_{\mathfrak{S}}^N/\mathcal{I}_{\mathfrak{S}}^n, F^\bullet) \in D_{\mathfrak{S}}^b(\mathfrak{X})_{< W-N}$ . Because  $D_{\mathfrak{S}}(\mathfrak{X})_{< w}$  is cocomplete, we have  $\varinjlim R\mathrm{Hom}_{\mathfrak{X}}(\mathcal{I}_{\mathfrak{S}}^N/\mathcal{I}_{\mathfrak{S}}^n, F^\bullet) \in D_{\mathfrak{S}}(\mathfrak{X})_{< W-N}$ . It follows that for any  $N > W - w$ ,

$$\beta_{\geq w} R\mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_X/\mathcal{I}_{\mathfrak{S}}^N, F^\bullet) \xrightarrow{\sim} \beta_{\geq w} R\Gamma_{\mathfrak{S}}(F^\bullet)$$

$\square$



The functor  $\beta_{\geq w} R\Gamma_{\mathfrak{S}}(\bullet)$  is right adjoint to the inclusion of the full subcategory  $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X})$ , because for  $F^\bullet \in D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}$  and  $G^\bullet \in D^b(\mathfrak{X})$  we have

$$\begin{aligned} R\mathrm{Hom}_{D(\mathfrak{X})}^\bullet(F^\bullet, G^\bullet) &\simeq R\mathrm{Hom}_{D_{\mathfrak{S}}(\mathfrak{X})}^\bullet(F^\bullet, R\Gamma_{\mathfrak{S}} G^\bullet) \\ &\simeq R\mathrm{Hom}_{D_{\mathfrak{S}}(\mathfrak{X})_{\geq w}}^\bullet(F^\bullet, \beta_{\geq w} R\Gamma_{\mathfrak{S}} G^\bullet) \end{aligned}$$

We also have that  $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X})_{\geq w}$  is right admissible for the same reason.

We pause here to discuss how Serre duality interacts with the structures we have defined. Let  $\omega_{\mathfrak{X}} := (\bigwedge^{\mathrm{top}} \mathfrak{g}) \otimes (\bigwedge^{\mathrm{top}} \Omega_X^1)$  be the canonical bundle of  $\mathfrak{X}$ . We have  $Rj^!(\omega_{\mathfrak{X}}) = \omega_{\mathfrak{S}}[-c]$  and  $Rj^!(\mathcal{O}_X) \simeq \det(\Omega_{\mathfrak{S}} \mathfrak{X})^\vee[-c]$  where  $c$  is the codimension of  $\tilde{S} \hookrightarrow X$ . If we use  $\omega_{\mathfrak{X}}[\mathrm{vdim} \mathfrak{X}]$  and  $\omega_{\mathfrak{S}}[\mathrm{vdim} \mathfrak{S}]$  as dualizing complexes on  $\mathfrak{X}$  and  $\mathfrak{S}$ , then we have the identity  $\mathbb{D}_{\mathfrak{S}} Lj^* F^\bullet \simeq Rj^! \mathbb{D}_{\mathfrak{X}} F^\bullet$ . From the remarks about Serre duality at the end of the last subsection, we have

$$\begin{aligned} F^\bullet \in D^b(\mathfrak{X})_{<w} &\Leftrightarrow \mathbb{D}_{\mathfrak{S}} Lj^* \mathbb{D}_{\mathfrak{X}} F^\bullet \in D^b(\mathfrak{S})_{<w} \\ &\Leftrightarrow Lj^* \mathbb{D}_{\mathfrak{X}} F^\bullet \in D^b(\mathfrak{S})_{\geq a+1-w} \end{aligned}$$

Thus we have  $\mathbb{D}_{\mathfrak{X}}(D^b(\mathfrak{X})_{\geq w}) = D^b(\mathfrak{X})_{<a+1-w}$

**Theorem 3.21.** *Let  $\mathbf{G}_w = D^b(\mathfrak{X})_{\geq w} \cap D^b(\mathfrak{X})_{<w}$ , and  $\kappa : \mathfrak{Z} \hookrightarrow \mathfrak{X}$  the closed immersion, then*

$$\mathbf{G}_w = \{F^\bullet \in D^b(\mathfrak{X}) \mid L\kappa^* F^\bullet \text{ supported in weights } [w, w + \eta]\}$$

where  $\eta$  is the weight of  $\det(\Omega_{\mathfrak{S}} \mathfrak{X})$ . There are semiorthogonal decompositions

$$D^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}, \mathbf{G}_w, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$$

And the restriction functor  $i^* : D^b(\mathfrak{X}) \rightarrow D^b(\mathfrak{Y})$  induces an equivalence  $\mathbf{G}_w \simeq D^b(\mathfrak{Y})$

*Proof.* We have  $Lj^* F^\bullet \in D^b(\mathfrak{S})_{\geq w}$  iff  $L\kappa^* F^\bullet \in D^b(\mathfrak{Z})_{\geq w}$  by definition. Also  $Rj^! F^\bullet \simeq Rj^! \mathcal{O}_X \otimes^L Lj^* F^\bullet \in D^b(\mathfrak{S})_{<w}$  iff  $\sigma^*(Rj^! \mathcal{O}_X) \otimes L\kappa^* F^\bullet \in D^b(\mathfrak{Z})_{<w}$ . Because  $Rj^! \mathcal{O}_X \simeq \det(\Omega_{\mathfrak{S}} \mathfrak{X})^\vee$ , we have  $F^\bullet \in D^b(\mathfrak{X})_{<w}$  iff  $L\kappa^* F^\bullet \in D^b(\mathfrak{Z})_{<w+\eta}$  where  $\eta$  is the weight of  $\Omega_{\mathfrak{S}} \mathfrak{X}$ .

The existence of a semiorthogonal decomposition is entirely a formal consequence of the previous discussion. We know that  $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X})$  is right admissible, and the adjunction between  $j_*$  and  $j^!$  along with the fact that  $j_* D^b(\mathfrak{S})_{\geq w}$  generates  $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}$  guarantees that  $D^b(\mathfrak{X})_{<w} = D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}^\perp$ .

By Serre duality  $D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$  is left admissible in  $D^b(\mathfrak{X})$ , and its left orthogonal can be similarly identified with  $D^b(\mathfrak{X})_{\geq w}$ . The existence of the claimed weak semiorthogonal decomposition now follows. Theorem 3.16 implies that  $i^* : \mathbf{G}_w \rightarrow D^b(\mathfrak{Y})$  is an equivalence of categories.  $\square$

One can explicitly define the inverse using the functors  $\beta_{\geq w}$  and  $\beta_{< w}$  on  $D_{\mathfrak{G}}^b(\mathfrak{X})$ . Given  $F^\bullet \in D^b(\mathfrak{Y})$ , choose a complex  $\tilde{F}^\bullet \in D^b(\mathfrak{X})$  such that  $\tilde{F}^\bullet|_{\mathfrak{Y}} \simeq F^\bullet$ . Now for  $N \gg 0$  take the mapping cone

$$\beta_{\geq w} R\text{Hom}_{\mathfrak{X}}(\mathcal{O}_X/\mathcal{I}_{\mathfrak{G}}^N, \tilde{F}^\bullet) = \beta_{\geq w} R\Gamma_{\mathfrak{G}} \tilde{F}^\bullet \rightarrow \tilde{F}^\bullet \rightarrow G^\bullet \dashrightarrow$$

So  $G^\bullet \in D^b(\mathfrak{X})_{< w}$ . By Serre duality the left adjoint of the inclusion  $D_{\mathfrak{G}}^b(\mathfrak{X})_{< w} \subset D^b(\mathfrak{X})_{< w}$  is  $\mathbb{D}_{\mathfrak{X}} \beta_{\geq \eta+1-w} R\Gamma_{\mathfrak{G}} \mathbb{D}_{\mathfrak{X}}$ , and this functor can be simplified using Lemma 3.20. We form the exact triangle

$$\tilde{G}^\bullet \rightarrow G^\bullet \rightarrow \beta_{< w}(G^\bullet \otimes^L \mathcal{O}_X/\mathcal{I}_{\mathfrak{G}}^N) \dashrightarrow$$

and  $\tilde{G}^\bullet \in \mathbf{G}_w$  is the unique object in  $\mathbf{G}_w$  mapping to  $F^\bullet$ .

#### 4. DERIVED EQUIVALENCES AND VARIATION OF GIT

We apply Theorem 2.2 to the derived categories of birational varieties obtained by a variation of GIT quotient. First we study the case where  $G = \mathbb{C}^*$ , in which the KN stratification is particularly easy to describe. Next we generalize this analysis to arbitrary variations of GIT, one consequence of which is the observation that if a smooth projective-over-affine variety  $X$  is equivariantly Calabi-Yau for the action of a torus, then the GIT quotients of any two generic linearizations are derived equivalent.

A normal projective variety  $X$  with linearized  $\mathbb{C}^*$  action is sometimes referred to as a birational cobordism between  $X//_{\mathcal{L}} G$  and  $X//_{\mathcal{L}(r)} G$  where  $\mathcal{L}(m)$  denotes the twist of  $\mathcal{L}$  by the character  $t \mapsto t^r$ . A priori this seems like a highly restrictive type of VGIT, but by Thaddeus' master space construction[18], any two spaces that are related by a general VGIT are related by a birational cobordism. We also have the weak converse due to Hu & Keel:

**Theorem 4.1** (Hu & Keel). *Let  $Y_1$  and  $Y_2$  be two birational projective varieties, then there is a birational cobordism  $X/\mathbb{C}^*$  between  $Y_1$  and  $Y_2$ . If  $Y_1$  and  $Y_2$  are smooth, then by equivariant resolution of singularities  $X$  can be chosen to be smooth.*

The GIT stratification for  $G = \mathbb{C}^*$  is very simple. If  $\mathcal{L}$  is chosen so that the GIT quotient is an orbifold, then the  $Z_\alpha$  are the connected components of the fixed locus  $X^G$ , and  $S_\alpha$  is either the ascending or descending manifold of  $Z_\alpha$ , depending on the weight of  $\mathcal{L}$  along  $Z_\alpha$ .

We will denote the tautological choice of 1-PS as  $\lambda^+$ , and we refer to “the weights” of a coherent sheaf at point in  $X^G$  as the weights with respect to this 1-PS. We define  $\mu_\alpha \in \mathbb{Z}$  to be the weight of  $\mathcal{L}|_{Z_\alpha}$ . If  $\mu_\alpha > 0$  (respectively  $\mu_\alpha < 0$ ) then the maximal destabilizing 1-PS of  $Z_\alpha$  is  $\lambda^+$  (respectively  $\lambda^-$ ). Thus we have

$$S_\alpha = \left\{ x \in X \left| \begin{array}{l} \lim_{t \rightarrow 0} t \cdot x \in Z_\alpha \text{ if } \mu_\alpha > 0 \\ \lim_{t \rightarrow 0} t^{-1} \cdot x \in Z_\alpha \text{ if } \mu_\alpha < 0 \end{array} \right. \right\}$$

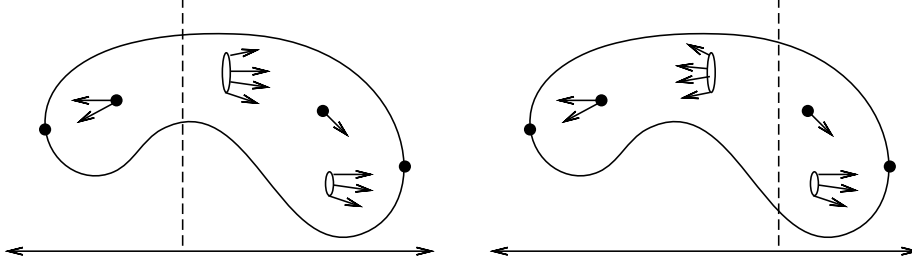


FIGURE 1. Schematic diagram for the fixed loci  $Z_\alpha$ .  $S_\alpha$  is the ascending or descending manifold of  $Z_\alpha$  depending on the sign of  $\mu_\alpha$ . As the moment fiber varies, the unstable strata  $S_\alpha$  flip over the critical sets  $Z_\alpha$ .

Next observe the weight decomposition under  $\lambda^+$

$$\Omega_X^1|_{Z_\alpha} \simeq \Omega_{Z_\alpha}^1 \oplus \mathcal{N}^+ \oplus \mathcal{N}^- \quad (8)$$

Then  $\Omega_{S_\alpha}^1|_{3^\alpha} = \Omega_{Z_\alpha}^1 \oplus \mathcal{N}^-$  if  $\mu_\alpha > 0$  and  $\Omega_{S_\alpha}^1|_{3^\alpha} = \Omega_{Z_\alpha}^1 \oplus \mathcal{N}^+$  if  $\mu_\alpha < 0$ , so we have

$$\eta_\alpha = \begin{cases} \text{weight of } \det \mathcal{N}^+|_{Z_\alpha} & \text{if } \mu_\alpha > 0 \\ -\text{weight of } \det \mathcal{N}^-|_{Z_\alpha} & \text{if } \mu_\alpha < 0 \end{cases} \quad (9)$$

There is a parallel interpretation of this in the symplectic category. A sufficiently large power of  $\mathcal{L}$  induces an equivariant projective embedding and thus a moment map  $\mu : X \rightarrow \mathbb{R}$  for the action of  $S^1 \subset \mathbb{C}^*$ . The semistable locus is the orbit of the zero fiber  $X^{ss} = G \cdot \mu^{-1}(0)$ . The reason for the collision of notation is that the fixed loci  $Z_\alpha$  are precisely the critical points of  $\mu$ , and the number  $\mu_\alpha$  is the value of the moment map on the critical set  $Z_\alpha$ .

Varying the linearization  $\mathcal{L}(r)$  by twisting by the character  $t \mapsto t^{-r}$  corresponds to shifting the moment map by  $-r$ , so the new zero fiber corresponds to what was previously the fiber  $\mu^{-1}(r)$ . For non-critical moment fibers the GIT quotient will be a DM stack, and the critical values of  $r$  are those for which  $\mu_\alpha = \text{weight of } \mathcal{L}(r)|_{Z_\alpha} = 0$  for some  $\alpha$ .

Say that as  $r$  increases it crosses a critical value for which  $\mu_\alpha = 0$ . The maximal destabilizing 1-PS  $\lambda_\alpha$  flips from  $\lambda^+$  to  $\lambda^-$ , and the unstable stratum  $S_\alpha$  flips from the ascending manifold of  $Z_\alpha$  to the descending manifold of  $Z_\alpha$ . In the decomposition (8), the normal bundle of  $S_\alpha$  changes from  $\mathcal{N}^+$  to  $\mathcal{N}^-$ , so applying  $\det$  to (8) and taking the weight gives

$$\text{weight of } \omega_X|_{Z_\alpha} = \eta_\alpha - \eta'_\alpha \quad (10)$$

Thus if  $\omega_X$  has weight 0 along  $Z_\alpha$ , the integer  $\eta_\alpha$  does not change as we cross the wall. The grade restriction window of Theorem 2.2 has the same width for the GIT quotient on either side of the wall, and it follows that the two GIT quotients are derived equivalent because they are identified with

the *same* subcategory  $\mathbf{G}_q$  of the equivariant derived category  $D^b(X/G)$ . We summarize this with the following

**Proposition 4.2.** *Let  $\mathcal{L}$  be a critical linearization of  $X/\mathbb{C}^*$ , and assume that  $Z_\alpha$  is the only critical set for which  $\mu_\alpha = 0$ . Let  $a$  be the weight of  $\omega_X|_{Z_\alpha}$ , and let  $\epsilon > 0$  be a small rational number.*

(1) *If  $a > 0$ , then there is a fully faithful embedding*

$$D^b(X//_{\mathcal{L}(\epsilon)}G) \subseteq D^b(X//_{\mathcal{L}(-\epsilon)}G)$$

(2) *If  $a = 0$ , then there is an equivalence*

$$D^b(X//_{\mathcal{L}(\epsilon)}G) \simeq D^b(X//_{\mathcal{L}(-\epsilon)}G)$$

(3) *If  $a < 0$ , then there is a fully faithful embedding*

$$D^b(X//_{\mathcal{L}(-\epsilon)}G) \subseteq D^b(X//_{\mathcal{L}(\epsilon)}G)$$

The analytic local model for a birational cobordism is the following

**Example 4.3.** Let  $Z$  be a smooth variety and let  $\mathcal{N} = \bigoplus \mathcal{N}_i$  be a  $\mathbb{Z}$ -graded locally free sheaf on  $Z$  with  $\mathcal{N}_0 = 0$ . Let  $X$  be the total of  $\mathcal{N}$  – it has a  $\mathbb{C}^*$  action induced by the grading. Because the only fixed locus is  $Z$  the underlying line bundle of the linearization is irrelevant, so we take the linearization  $\mathcal{O}_X(r)$ .

If  $r > 0$  then the unstable locus is  $\mathcal{N}_- \subset X$  where  $\mathcal{N}_-$  is the sum of negative weight spaces of  $\mathcal{N}$ , and if  $r < 0$  then the unstable locus is  $\mathcal{N}_+$  (we are abusing notation slightly by using the same notation for the sheaf and its total space). We will borrow the notation of Thaddeus [18] and write  $X/\pm = (X \setminus \mathcal{N}_\mp)/\mathbb{C}^*$ .

Inside  $X/\pm$  we have  $\mathcal{N}_\pm/\pm \simeq \mathbb{P}(\mathcal{N}_\pm)$ , where we are still working with quotient stacks, so the notation  $\mathbb{P}(\mathcal{N}_\pm)$  denotes the weighted projective bundle associated to the graded locally free sheaf  $\mathcal{N}_\pm$ . If  $\pi_\pm : \mathbb{P}(\mathcal{N}_\pm) \rightarrow Z$  is the projection, then  $X/\pm$  is the total space of the vector bundle  $\pi_\pm^* \mathcal{N}_\mp(-1)$ . We have the common resolution

$$\begin{array}{ccc} & \mathcal{O}_{\mathbb{P}(\mathcal{N}_-) \times_S \mathbb{P}(\mathcal{N}_+)}(-1, -1) & \\ \swarrow & & \searrow \\ \pi_+^* \mathcal{N}_-(-1) & & \pi_-^* \mathcal{N}_+(-1) \end{array}$$

Let  $\pi : X \rightarrow Z$  be the projection, then the canonical bundle is  $\omega_X = \pi^*(\omega_Z \otimes \det(\mathcal{N}_+)^{\vee} \otimes \det(\mathcal{N}_-)^{\vee})$ , so the weight of  $\omega_X|_Z$  is  $\sum i \operatorname{rank}(\mathcal{N}_i)$ . In the special case of a flop, Proposition 4.2 says

$$\text{if } \sum i \operatorname{rank}(\mathcal{N}_i) = 0, \text{ then } D^b(\pi_+^* \mathcal{N}_-(-1)) \simeq D^b(\pi_-^* \mathcal{N}_+(-1))$$

**4.1. General variation of GIT quotient.** We will generalize the analysis of a birational cobordism to an arbitrary variation of GIT quotient. Until this point we have taken the KN stratification as given, but now we must recall its definition and basic properties as described in [9].

Let  $\mathrm{NS}^G(X)_{\mathbb{R}}$  denote the group of equivariant line bundles up to homological equivalence, tensored with  $\mathbb{R}$ . For any  $\mathcal{L} \in \mathrm{NS}^G(X)_{\mathbb{R}}$  one defines a stability function on  $X$

$$M^{\mathcal{L}}(x) := \max \left\{ \frac{\mathrm{weight}_{\lambda} \mathcal{L}_y}{|\lambda|} \mid \lambda \text{ s.t. } y = \lim_{t \rightarrow 0} \lambda(t) \cdot x \text{ exists} \right\}$$

$M^{\mathcal{L}}(\bullet)$  is upper semi-continuous, and  $M^{\bullet}(x)$  is lower convex and thus continuous on  $\mathrm{NS}^G(X)_{\mathbb{R}}$  for a fixed  $x$ . A point  $x \in X$  is semistable if  $M^{\mathcal{L}}(x) \leq 0$ , stable if  $M^{\mathcal{L}}(x) < 0$ , strictly semistable if  $M^{\mathcal{L}}(x) = 0$  and unstable if  $M^{\mathcal{L}}(x) > 0$ .

Now let  $\mathcal{P}$  denote the set of pairs  $(\lambda, Z)$  where  $\lambda$  is a 1-PS of  $G$  and  $Z$  is a connected component of  $X^{\lambda}$ , modulo the action of  $G$ . The set of  $Z$  appearing in such pairs is finite up to the action of  $G$ . Define the numerical invariant  $\mu^{\mathcal{L}}(\lambda, Z) = \frac{1}{|\lambda|} \mathrm{weight}_{\lambda} \mathcal{L}|_Z \in \mathbb{R}$ . We can construct the KN stratification iteratively by taking a  $(\lambda_{\alpha}, Z_{\alpha}) \in \mathcal{P}'$  maximizing  $\mu^{\mathcal{L}}$ , considering the open subset  $Z_{\alpha}^{\circ} \subset Z_{\alpha}$  not intersecting any higher strata, and defining  $S_{\alpha}$  to be the orbit of the attracting set of  $Z_{\alpha}^{\circ}$  under  $\lambda_{\alpha}$ .

The  $G$ -ample cone  $\mathcal{C}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$  has a finite decomposition into convex conical chambers separated by hyperplanes – the interior of a chamber is where  $M^{\mathcal{L}}(x) \neq 0$  for all  $x \in X$ , so  $\mathfrak{X}^{ss}(\mathcal{L}) = \mathfrak{X}^s(\mathcal{L})$ . We will focus on a single wall-crossing:  $\mathcal{L}_0$  will be a  $G$ -ample line bundle lying on a wall such that for  $\epsilon$  sufficiently small  $\mathcal{L}_{\pm} := \mathcal{L}_0 \pm \epsilon \mathcal{L}'$  both lie in the interior of chambers.

By continuity of the function  $M^{\bullet}(x)$  on  $\mathrm{NS}^G(X)_{\mathbb{R}}$ , all of the stable and unstable points of  $\mathfrak{X}^s(\mathcal{L}_0)$  will remain so for  $\mathcal{L}_{\pm}$ . Only points in the strictly semistable locus,  $\mathfrak{X}^{sss}(\mathcal{L}_0) = \{x \in \mathfrak{X} \mid M^{\mathcal{L}}(x) = 0\} \subset \mathfrak{X}$ , change from being stable to unstable as one crosses the wall.

In fact  $\mathfrak{X}^{us}(\mathcal{L}_0)$  is a union of KN strata for  $\mathfrak{X}^{us}(\mathcal{L}_+)$ , and symmetrically it can be written as a union of KN strata for  $\mathfrak{X}^{us}(\mathcal{L}_-)$ . [9] Thus we can write  $\mathfrak{X}^{ss}(\mathcal{L}_0)$  in two ways

$$\mathfrak{X}^{ss}(\mathcal{L}_0) = \mathfrak{S}_1^{\pm} \cup \dots \cup \mathfrak{S}_{m_{\pm}}^{\pm} \cup \mathfrak{X}^{ss}(\mathcal{L}_{\pm}) \quad (11)$$

Where  $\mathfrak{S}_i^{\pm}$  are the KN strata of  $\mathfrak{X}^{us}(\mathcal{L}_{\pm})$  lying in  $\mathfrak{X}^{ss}(\mathcal{L}_0)$ .

**Definition 4.4.** A wall crossing  $\mathcal{L}_{\pm} = \mathcal{L}_0 \pm \epsilon \mathcal{L}'$  will be called *balanced* if  $m_+ = m_-$  and  $\mathfrak{Z}_i^+ = \mathfrak{Z}_i^-$  under the decomposition (11).

By the construction of the strata outlined above, there is a finite collection of locally closed  $Z_i \subset X$  and one parameter subgroups  $\lambda_i$  fixing  $Z_i$  such that  $G \cdot Z_i / G$  are simultaneously the attractors for the KN strata of both  $\mathfrak{X}^{ss}(\mathcal{L}_{\pm})$  and such that the  $\lambda_i^{\pm 1}$  are the maximal destabilizing 1-PS's.

**Proposition 4.5.** *Let a reductive  $G$  act on a projective-over-affine variety  $X$ . Let  $\mathcal{L}_0$  be a  $G$ -ample line bundle on a wall, and define  $\mathcal{L}_\pm = \mathcal{L}_0 \pm \epsilon \mathcal{L}'$  for some other line bundle  $\mathcal{L}'$ . Assume that*

- *for  $\epsilon$  sufficiently small,  $\mathfrak{X}^{ss}(\mathcal{L}_\pm) = \mathfrak{X}^s(\mathcal{L}_\pm) \neq \emptyset$ ,*
- *the wall crossing  $\mathcal{L}_\pm$  is balanced, and*
- *for all  $Z_i$  in  $\mathfrak{X}^{ss}(\mathcal{L}_0)$ ,  $(\omega_{\mathfrak{X}})|_{Z_i}$  has weight 0 with respect to  $\lambda_i$*

*then  $D^b(\mathfrak{X}^{ss}(\mathcal{L}_+)) \simeq D^b(\mathfrak{X}^{ss}(\mathcal{L}_-))$ .*

**Remark 4.6.** Full embeddings analagous to those of Proposition 4.2 apply when the weights of  $(\omega_{\mathfrak{X}})|_{Z_i}$  with respect to  $\lambda_i$  are either all negative or all positive.

*Proof.* The proof is an immediate application of Theorem 2.2 to the open substack  $\mathfrak{X}^s(\mathcal{L}_\pm) \subset \mathfrak{X}^{ss}(\mathcal{L}_0)$  whose complement admits the KN stratification (11). Because the wall crossing is balanced,  $Z_i^+ = Z_i^-$  and  $\lambda_i^-(t) = \lambda_i^+(t^{-1})$ , and the condition on  $\omega_{\mathfrak{X}}$  implies that  $\eta_i^+ = \eta_i^-$ . So Theorem 2.2 identifies the category  $\mathbf{G}_q \subset D^b(\mathfrak{X}^{ss}(\mathcal{L}_0))$  with both  $D^b(\mathfrak{X}^s(\mathcal{L}_-))$  and  $D^b(\mathfrak{X}^s(\mathcal{L}_+))$ .  $\square$

**Example 4.7.** Dolgachev and Hu study wall crossings which they call *truly faithful*, meaning that the identity component of the stabilizer of a point with closed orbit in  $\mathfrak{X}^{ss}(\mathcal{L}_0)$  is  $\mathbb{C}^*$ . They show that every truly faithful wall is balanced.[9, Lemma 4.2.3]

Dolgachev and Hu also show that for the action of a torus  $T$ , there are no codimension 0 walls and all codimension 1 walls are truly faithful. Thus any two chambers in  $\mathcal{C}^T(X)$  can be connected by a finite sequence of balanced wall crossings, and we have

**Corollary 4.8.** *Let  $X$  be a projective-over-affine variety with an action of a torus  $T$ . Assume  $X$  is equivariantly Calabi-Yau in the sense that  $\omega_X \simeq \mathcal{O}_X$  as an equivariant  $\mathcal{O}_X$ -module. If  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are  $G$ -ample line bundles such that  $\mathfrak{X}^s(\mathcal{L}_i) = \mathfrak{X}^{ss}(\mathcal{L}_i)$ , then  $D^b(\mathfrak{X}^s(\mathcal{L}_0)) \simeq D^b(\mathfrak{X}^s(\mathcal{L}_1))$ .*

A compact projective manifold with a non-trivial  $\mathbb{C}^*$  action is never equivariantly Calabi-Yau, but Corollary 4.8 applies to a large class of non compact examples. The simplest are linear representations  $V$  of  $T$  such that  $\det V$  is trivial. More generally we have

**Example 4.9.** Let  $T$  act on a smooth projective Fano variety  $X$ , and let  $\mathcal{E}$  be an equivariant ample locally free sheaf such that  $\det \mathcal{E} \simeq \omega_X^\vee$ . Then the total space of the dual vector bundle  $Y = \mathrm{Spec}_X(S^*\mathcal{E})$  is equivariantly Calabi-Yau and the canonical map  $Y \rightarrow \mathrm{Spec}(\Gamma(X, S^*\mathcal{E}))$  is projective, so  $Y$  is projective over affine and by Corollary 4.8 any two generic GIT quotients  $Y//T$  are derived equivalent.

When  $G$  is non-abelian, the chamber structure of  $\mathcal{C}^G(X)$  can be more complicated. There can be walls of codimension 0, meaning open regions in the interior of  $\mathcal{C}^G(X)$  where  $\mathfrak{X}^s \neq \mathfrak{X}^{ss}$ , and not all walls are truly faithful.[9]

However, the property of a wall crossing being balanced is weaker than being truly faithful, and can sometimes be verified directly.

**Example 4.10.** Choose  $k < N$  and let  $V$  be a  $k$ -dimensional vector space. Consider the action of  $G = GL(V)$  on  $X = T^* \text{Hom}(V, \mathbb{C}^N) = \text{Hom}(V, \mathbb{C}^N) \times \text{Hom}(\mathbb{C}^N, V)$ . A 1-PS  $\lambda : \mathbb{C}^* \rightarrow G$  corresponds to a choice of weight decomposition  $V \simeq \bigoplus V_k$  under  $\lambda$ . A point  $(a, b)$  has a limit under  $\lambda$  iff

$$V_{>0} \subset \ker(a) \quad \text{and} \quad \text{im}(b) \subset V_{\geq 0}$$

in which case the limit  $(a_0, b_0)$  is the projection onto  $V_0 \subset V$ . There are only two nontrivial characters up to rational equivalence,  $\det^\pm$ . A point  $(a, b)$  is semistable iff any 1-PS for which  $\lambda(t) \cdot (a, b)$  has a limit as  $t \rightarrow 0$  has nonpositive pairing with the chosen character.

If we linearize with respect to  $\det$ , then  $(a, b) \in X$  is semistable iff  $a$  is injective. The unstable strata are indexed by  $i = 1, \dots, k$

$$\tilde{Z}_i = \{(a, b) \in X \mid V = \text{im}(b) \oplus \ker(a) \text{ and } \dim(\ker(a)) = i\} \quad (12)$$

The distinguished 1-PS  $\lambda_i^+$  for a point  $(a, b) \in \tilde{Z}_i$  acts with weight 0 on  $\text{im}(b)$  and with weight 1 on  $\ker(a)$ . If instead we linearize with respect to  $\det^{-1}$ , then  $(a, b)$  is semistable iff  $b$  is surjective, and the strata  $\tilde{Z}_i$  are exactly the same as (12). So this is a balanced wall crossing with  $\mathcal{L}_0 = \mathcal{O}_X$  and  $\mathcal{L}' = \mathcal{O}_X(\det)$ .

Finally, the variety  $X$  is equivariantly Calabi-Yau, so  $\omega_X$  has weight 0 with respect to all  $\lambda_i^\pm$ . By Proposition 4.5 the two GIT quotients are derived-equivalent.

Let  $\mathbb{G}(k, N)$  be the Grassmannian parametrizing  $k$ -dimensional subspaces  $V \subset \mathbb{C}^N$ , and let  $0 \rightarrow U(k, N) \rightarrow \mathcal{O}^N \rightarrow Q(k, N) \rightarrow 0$  be the tautological sequence of vector bundles on  $\mathbb{G}(k, N)$ . Then  $\mathfrak{X}^{ss}(\det)$  is the total space of  $U(k, N)^N$ , and  $\mathfrak{X}^{ss}(\det^{-1})$  is the total space of  $(Q(N - k, N)^\vee)^N$  over  $\mathbb{G}(N - k, N)$ . Thus we have established an equivalence of derived categories

$$\text{D}^b(U(k, N)^N) \simeq \text{D}^b((Q(N - k, N)^\vee)^N)$$

The astute reader will observe that these two varieties are in fact isomorphic, but the derived equivalences we have constructed are natural in the sense that they generalize to families. Specifically, if  $\mathcal{E}$  is an  $N$ -dimensional vector bundle over a smooth variety  $Y$ , then the two GIT quotients of the total space of  $\underline{\text{Hom}}(\mathcal{O}_Y \otimes V, \mathcal{E}) \oplus \underline{\text{Hom}}(\mathcal{E}, \mathcal{O}_Y \otimes V)$  by  $GL(V)$  will have equivalent derived categories.

**Remark 4.11.** This example is similar to the generalized Mukai flops of [7]. The difference is that we are not restricting to the hyperkähler moment fiber  $\{ba = 0\}$ . The surjectivity theorem cannot be applied directly to the GIT quotient of this singular variety, but in the next section we will explore some applications to abelian hyperkähler reduction.

## 5. CONSEQUENCES FOR COMPLETE INTERSECTIONS, MATRIX FACTORIZATIONS, AND HYPERKÄHLER REDUCTION

In this section I remark that Theorem 1.1 extends to complete intersections in a smooth  $X/G$  for purely formal reasons, where by complete intersection I mean one defined by global invariant functions on  $X/G$ .

In this section I will use derived Morita theory ([3],[12]), and so I will switch to a notation more common in that subject.  $\mathrm{QC}(\mathfrak{X})$  will denote the unbounded derived category of quasicoherent sheaves on a perfect stack  $\mathfrak{X}$ , and  $\mathrm{Perf}(\mathfrak{X})$  will denote the category of perfect complexes, i.e. the compact objects of  $\mathrm{QC}(\mathfrak{X})$ . All of the stacks we use are global quotients of quasiprojective varieties, so  $\mathrm{Perf}(\mathfrak{X})$  are just the objects of  $\mathrm{QC}(\mathfrak{X})$  which are equivalent to a complex of vector bundles.

Now let  $\mathfrak{X} = X/G$  as in the rest of this paper. Assume we have a map  $f : \mathfrak{X} \rightarrow B$  where  $B$  is a quasiprojective scheme. The restriction  $i^* : \mathrm{Perf}(\mathfrak{X}) \rightarrow \mathrm{Perf}(\mathfrak{X}^{ss})$  is a dg- $\otimes$  functor, and in particular it is a functor of module categories over the monoidal dg-category  $\mathrm{Perf}(B)^{\otimes}$ .

The subcategory  $\mathbf{G}_q$  used to construct the splitting in Theorem 1.1 is defined using conditions on the weights of various 1-PS's of the isotropy groups of  $\mathfrak{X}$ , so tensoring by a vector bundle  $f^*V$  from  $B$  preserves the subcategory  $\mathbf{G}_q$ . It follows that the splitting constructed in Theorem 1.1 is a splitting as modules over  $\mathrm{Perf}(B)$ . Thus for any point  $b \in B$  we have a split surjection

$$\mathrm{Fun}_{\mathrm{Perf}(B)}(\mathrm{Perf}(\{b\}), \mathrm{Perf}(\mathfrak{X})) \xrightleftharpoons[i^*]{} \mathrm{Fun}_{\mathrm{Perf}(B)}(\mathrm{Perf}(\{b\}), \mathrm{Perf}(\mathfrak{X}^{ss}))$$

Using Morita theory, both functor categories correspond to full subcategories of  $\mathrm{QC}((\bullet)_b)$ , where  $(\bullet)_b$  denotes the derived fiber  $(\bullet) \times_B^L \{b\}$ . Explicitly,  $\mathrm{Fun}_{\mathrm{Perf}(B)}(\mathrm{Perf}(\{b\}), \mathrm{Perf}(\mathfrak{X}))$  is equivalent to the full dg-subcategory of  $\mathrm{QC}((\mathfrak{X})_b)$  consisting of complexes of sheaves whose pushforward to  $\mathfrak{X}$  is perfect. Because  $\mathfrak{X}$  is smooth, and  $\mathcal{O}_{(\mathfrak{X})_b}$  is coherent over  $\mathcal{O}_{\mathfrak{X}}$ , this is precisely the derived category of coherent sheaves  $D^b(\mathrm{Coh}((\mathfrak{X})_b))$ . The same analysis applied to the tensor product  $\mathrm{Perf}(\{b\}) \otimes_{\mathrm{Perf}(B)} \mathrm{Perf}(\mathfrak{X})$  yields a splitting for the category of perfect complexes.

**Corollary 5.1.** *Given a map  $f : \mathfrak{X} \rightarrow B$  and a point  $b \in B$ , the splitting of Theorem 1.1 induces splittings of the natural restriction functors*

$$\begin{aligned} D^b(\mathrm{Coh}((\mathfrak{X})_b)) &\xrightleftharpoons[i^*]{} D^b(\mathrm{Coh}((\mathfrak{X}^{ss})_b)) \\ \mathrm{Perf}((\mathfrak{X})_b) &\xrightleftharpoons[i^*]{} \mathrm{Perf}((\mathfrak{X}^{ss})_b) \end{aligned}$$

*In the particular case of a complete intersection one has  $B = \mathbb{A}^r$ ,  $b = 0 \in B$ , and the derived fiber agrees with the non-derived fiber.*

As a special case of Corollary 5.1, one obtains equivalences of categories of matrix factorizations in the form of derived categories of singularities.



Namely, if  $W : \mathfrak{X} \rightarrow \mathbb{C}$  is a function, a “potential” in the language of mirror symmetry, then the category of matrix factorizations corresponding to  $W$  is

$$\mathrm{MF}(\mathfrak{X}, W) \simeq \mathrm{D}_{\mathrm{sing}}^b(W^{-1}(0)) = \mathrm{D}^b(\mathrm{Coh}(W^{-1}(0))) / \mathrm{Perf}(W^{-1}(0))$$

From Corollary 5.1 the restriction functor  $\mathrm{MF}(\mathfrak{X}, W) \rightarrow \mathrm{MF}(\mathfrak{X}, W)$  splits. In particular, if two GIT quotients  $\mathrm{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_1))$  and  $\mathrm{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_2))$  can be identified with the same subcategory of  $\mathrm{Perf}(\mathfrak{X})$  as in Proposition 4.2, then the corresponding subcategories of matrix factorizations are equivalent

$$\mathrm{MF}(\mathfrak{X}^{ss}(\mathcal{L}_1), W|_{\mathfrak{X}^{ss}(\mathcal{L}_1)}) \simeq \mathrm{MF}(\mathfrak{X}^{ss}(\mathcal{L}_2), W|_{\mathfrak{X}^{ss}(\mathcal{L}_2)})$$

Corollary 5.1 also applies to the context of hyperkähler reduction. Let  $T$  be a torus, or any group whose connected component is a torus, and consider a Hamiltonian action of  $T$  on a hyperkähler variety  $X$  with algebraic moment map  $\mu : X/T \rightarrow \mathfrak{t}^\vee$ . One forms the hyperkähler quotient by choosing a linearization on  $X/T$  and defining  $X///T = \mu^{-1}(0) \cap \mathfrak{X}^{ss}$ . Thus we are in the setting of Corollary 5.1.

**Corollary 5.2.** *Let  $T$  be an extension of a finite group by a torus. Let  $T$  act on a hyperkähler variety  $X$  with algebraic moment map  $\mu : X \rightarrow \mathfrak{t}^\vee$ . Then the restriction functors*

$$\begin{aligned} \mathrm{D}(\mathrm{Coh}(\mu^{-1}(0)/T)) &\rightarrow \mathrm{D}(\mathrm{Coh}(\mu^{-1}(0)^{ss}/T)) \\ \mathrm{Perf}(\mathrm{Coh}(\mu^{-1}(0)/T)) &\rightarrow \mathrm{Perf}(\mathrm{Coh}(\mu^{-1}(0)^{ss}/T)) \end{aligned}$$

*both split.*

This splitting does not give as direct a relationship between  $\mathrm{D}^b(X/T)$  and  $\mathrm{D}^b(X///T)$  as Theorem 2.2 does for the usual GIT quotient, but it is enough for some applications, for instance

**Corollary 5.3.** *Let  $X$  be a projective-over-affine hyperkähler variety with a Hamiltonian action of a torus  $T$ . Then the hyperkähler quotients with respect to any two generic linearization  $\mathcal{L}_1, \mathcal{L}_2$  are derived equivalent.*

*Proof.* By Corollary 4.8 all  $\mathfrak{X}^{ss}(\mathcal{L})$  for generic  $\mathcal{L}$  will be derived equivalent. In particular there is a finite sequence of wall crossings  $\mathrm{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_+)) \rightarrow \mathrm{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_0)) \leftarrow \mathrm{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_-))$  identifying each GIT quotient with the same subcategory. By Corollary 5.2 these splittings descend to  $\mu^{-1}(0)$ , giving equivalences of both  $\mathrm{D}^b(\mathrm{Coh}(\bullet))$  and  $\mathrm{Perf}(\bullet)$  for the hyperkähler reductions.  $\square$

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